A UNIVERSAL STRANGE ATTRACTOR UNDERLYING THE QUASIPERIODIC TRANSITION TO CHAOS

David K. UMBERGER [1], J. Doyne FARMER

Center for Nonlinear Studies, MS B258, Los Alamos National Laboratory, Los Alamos, NM 87545, USA

and

Indubala I. SATIJA

Bartol Research Foundation, Newark, DE 19716, USA

Received 19 November 1985; accepted for publication 19 December 1985

We use a Monte Carlo approach to study the universal properties associated with the breakdown of two-torus attractors for arbitrary winding numbers. We demonstrate that the renormalization equations have a universal strange attractor, compute its critical exponents, and discuss its structure. The fractal dimension of this attractor is 1.8 ± 0.1.

One of the most common routes to chaos is the quasiperiodic transition, in which regular motion with two incommensurate frequencies becomes aperiodic. The universal properties of this transition are now well understood in some special cases, such as when the ratio of the frequencies, or winding number, is equal to the golden mean [1,2]. It is desirable, however, to have a more general theory that encompasses arbitrary winding numbers. Rand et al. [2] presented a set of renormalization equations which made this possible in principle, but they were unable to solve these equations (numerically or otherwise) except in a few special cases corresponding to the golden mean and its close relatives.

In a recent paper, Farmer and Satija [3] extended the renormalization transformation of Rand et al. into a two-dimensional domain. This causes the transformation to have stable attractors. Since this eliminates the need to linearize the equations, it becomes possible to study *nonlocal* properties. Their results suggested the existence of a strange attractor of the renormalization transformation, as conjectured by Rand et al. [2]. Due to numerical difficulties, however, it was impossible to study the details of the fractal structure.

In this paper we describe how the renormalization transformation introduced by Rand et al. [2] can be implemented with a Monte Carlo technique, allowing us to study arbitrary winding numbers. As they conjectured, we find a universal strange attractor. We explicitly demonstrate its ergodic properties and fractal structure, and compute a fractal dimension of 1.8 ± 0.1. We also compute the universal exponents, show that they agree with those previously found in ref. [3], and indicate that there is an approximate connection to a different set of exponents computed by Jensen et al. [4]. To our knowledge, we are the first to demonstrate that the renormalization transformation can have strange attractors.

In studying the quasiperiodic transition to chaos it is convenient to study circle maps, which can be thought of as representing the Poincaré map of the continuous problem. We employ two different maps in our work, the most famous of which is

$$x_{i+1} = x_i - (k/2\pi)\sin(2\pi x_i) + \omega\ , \tag{1}$$

where x is taken modulo 1, k is the nonlinearity pa-

[1] Also at: Department of Physics, University of Arkansas, Fayetteville, AR 72701, USA.

rameter, and ω is the parameter that determines the winding number ρ, defined by

$$\rho = \lim_{n\to\infty} [f^n(x) - x]/n .$$

For $k < 1$, the trajectories generated by this map are stably periodic (phase locked) when ρ is rational, and quasiperiodic when ρ is irrational. For $k > 1$ the quasiperiodic orbits are unstable and there exists a chaotic orbit in every neighborhood of the quasiperiodic orbit. Therefore, $k = 1$ is a special point where a transition from quasiperiodicity to chaos takes place.

The idea behind the renormalization schemes of refs. [1,2] is to approximate a quasiperiodic orbit by a sequence of periodic orbits. Following a method of Greene [5], this is done by expanding the winding number of the quasiperiodic motion in terms of continued fractions, i.e.,

$$\rho = \cfrac{1}{n_1 + \cfrac{1}{n_2 + \ldots}} ,$$

and truncating at successive terms to get a sequence of rationals. Alternative renormalization schemes based on a Farey tree expansion have also been proposed [6–8]. The continued-fraction expansion of a typical winding number does not terminate, but consists of random entries which can be found by iterating the Gauss map, $\rho_k = \rho_{k-1} - [1/\rho_{k-1}]$, where $[x]$ denotes the integer part of x. The Gauss map has chaotic solutions for almost all irrationals, with the invariant probability density

$$p(\rho) = [\log 2\,(1+\rho)]^{-1} . \qquad (2)$$

The chaotic behavior of the Gauss map implies that in the typical case, the renormalization equations do not have a fixed point, but instead generate a chaotic sequence of functions. For the subcritical case ($k < 1$) the strange attractor describing the resulting sequence of functions *is* the graph of the Gauss map, but in the critical case there is no longer a one-to-one correspondence between ω and ρ and things are not so simple [2,3]. For a given circle map f with winding number ρ, let $p_1/q_1, p_2/q_2, \ldots, p_j/q_j, \ldots$, be the sequence of rational approximants obtained by successive truncations of the continued-fraction expansion of ρ. As shown in ref. [2], in principle, an arbitrary winding number can be studied by constructing a sequence of renormalized circle maps

$$\xi_j(x) = \beta_j f^{q_j}(x/\beta_j) ,$$
$$\eta_j(x) = \beta_j f^{q_j+q_{j-1}}(x/\beta_j) . \qquad (3)$$

The scale factor β_j is chosen to satisfy the condition $\xi_j = 1 + \eta_j(0)$, which assures that the pair (ξ_j, η_j) taken together form a circle map. The domain of ξ_j is $(\eta_j(0), 0)$ and the domain of η_j is $(0, \xi_j(0))$. In principle, renormalization could be done for an arbitrary winding number by repeating this transformation indefinitely. Universal numbers could then be computed by taking a "time average" of the scale factors generated at each step. In practice this will not work, since it is impossible to know the winding number of the initial function f to arbitrary precision. After several iterations the true winding number of the renormalized function is no longer correct, and numerical errors occur.

In order to circumvent this problem we use a Monte Carlo approach, which amounts to taking an ensemble average rather than a time average. We pick a winding number at random according to eq. (2), and renormalize as described above until we detect a loss of precision. We then pick another winding number and repeat the process. Universal properties can be extracted by selecting out renormalised functions f_j with j in a range such that universal behavior is approached while at the same time numerical errors are not significant. Since the renormalization transformation is ergodic, we can compute average properties of the universal case by averaging over the functions that satisfy the above criteria. We are thus averaging over an ensemble of solutions of the renormalization transformation, picking out only those that have been iterated long enough that their properties are sufficiently close to universal.

We will now give some more details about how this is actually done. First we randomly generate a value of ρ weighted according to the probability distribution of eq. (2). Then, using Newton's method, we generate the superstable ω values ω_j corresponding to each rational approximant of ρ, until $|\omega_j - \omega_{j+1}| < 10^{-9}$. To determine f for eq. (3), we set $\omega_* = \omega_{j+1}$ and use $\omega = \omega_*$ in eq. (1). We then generate a sequence $(\xi, \eta)_j$ using eq. (3), stopping when $|\omega_j - \omega_*| < 10^{-8}$, which we found was a good criterion for the loss of numerical precision. In order to ensure that all the functions in our ensemble are sufficiently close to universal behavior, we rejected those for which

$$|(\omega_1 - \omega_0)/(\omega_{j+1} - \omega_j)| < 10^3 .$$

These criteria were arrived at after extensive numerical testing. Most of the computations were made using 14 digits of precision, but as a check many of them were repeated with 28 digits.

The universal functions $(\xi, \eta)_j$ generated in this manner look like a random sequence of functions. The important question is: What is the subset of the function space in which they wander? In particular, do they lie on an attractor, and if so, what is its dimension? To investigate this question, we projected the function space onto two dimensions by plotting $\xi_{j+1}(0)$ against $\xi_j(0)$ to produce a picture of the attractor. In the subcritical case the attractor is just the graph of the Gauss map, as shown in fig. 1. Each branch in the attractor corresponds to a given leading entry n in the continued-fraction expansion. The rightmost branch contains all the functions having $n = 1$, followed by a branch corresponding to functions with $n = 2$, and so on.

The critical case is more interesting. As you can see from fig. 2, the attractor gives a strong appearance of fractal structure, suggesting that it is a strange attractor. This impression is strengthened by plotting only those points with $n = 1$ or $n = 2$ in their continued-fraction expansion. The result, shown in fig. 3, consist of two strips, each with the appearance of the cartesian product of a Cantor set and an interval. Similarly, plotting points with $n = 3$ as well yields 3 such strips. As more strips are plotted these strips begin to overlap, presumably due to projection effects. This strongly suggests that the attractor consists of an infinite number of strips, each of which has a well-defined Cantor set structure.

An interesting feature of this attractor is that the

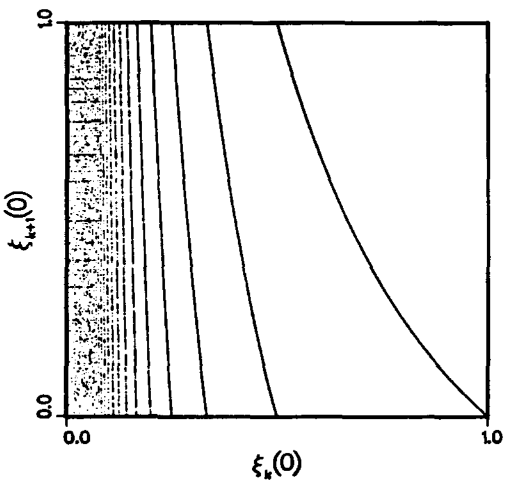

Fig. 1. A two-dimensional projection of the universal attractor in the subcritical case, made by plotting $\xi_{k+1}(0)$ versus $\xi_k(0)$.

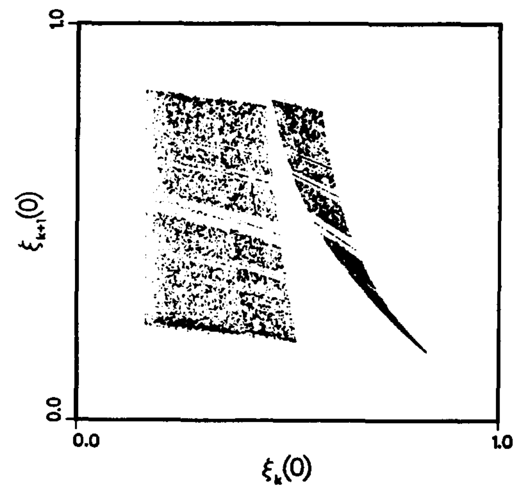

Fig. 2. A two-dimensional projection of the universal attractor for the critical case, as in fig. 1.

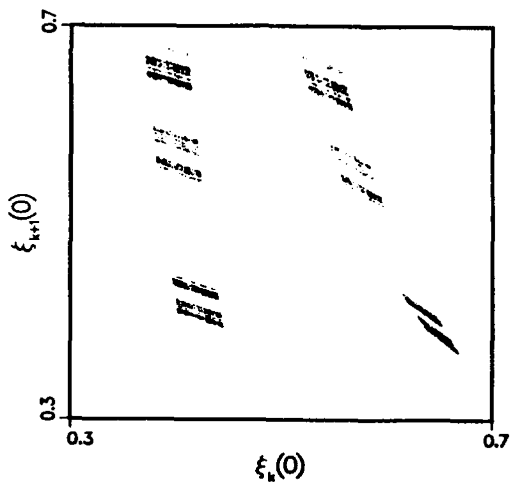

Fig. 3. Life fig. 2 except winding numbers with only a 1 or 2 in their continued-fraction expansions are used.

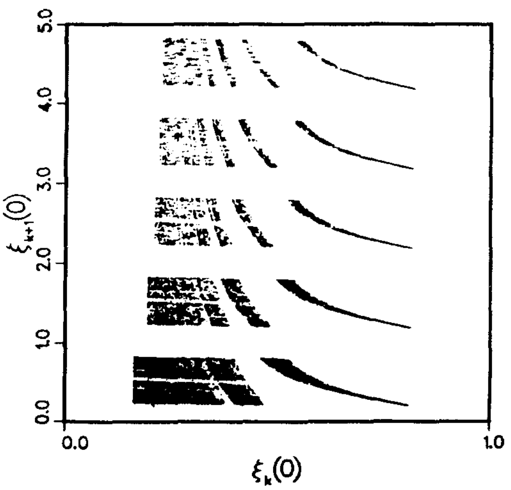

Fig. 4. Similar to fig. 2, except that functions whose leading entry in the continued-fraction expansion are 1, 2, ..., 5, are plotted separately and then stacked together on top of each other.

strips seen in figs. 2 and 3 do *not* correspond to the branches of the Gauss map. For the Gauss map appearance in a given branch is determined by the value of n_i but for the critical case there is no such correspondence. To demonstrate this graphically, we plotted the data in a manner similar to figs. 1 and 2 except that we added $n_i - 1$ to the vertical coordinate. For the subcritical case the resulting plot is simply the graph of the function $1/\rho$. For the critical case, however, the basic picture is repeated for every n_i, as shown in fig. 4. The structure of this set is intriguing and deserves further research.

We conjecture that the gaps in the attractor are a consequence of the mode locking of the critical circle map. A numerical computation of the fractal dimension of the attractor gives the value 1.8 ± 0.1. We interpret this dimension to be equal to one plus the fractal dimension of the Cantor set along the vertical direction on the attractor. Within numerical accuracy this dimension is equal to the fractal dimension of the complementary set to the devil stair case in the circle map as obtained by Jensen et al. [4]. This is consistent with our conjecture of the origin of the gaps in the attractor.

The critical exponents $\bar{\delta}$ and $\bar{\alpha}$ are computed by averaging the scaling factors at each step of renormalization. We define

$$\delta_j = (\omega_j - \omega_{j-1})/(\omega_{j+1} - \omega_j)\,, \quad \alpha_j = \beta_{j+1}/\beta_j\,,$$

$$\bar{\delta} = \lim_{N\to\infty} \left|\prod_{j=1}^{N} \delta_j\right|^{1/N}, \quad \bar{\alpha} = \lim_{N\to\infty} \left|\prod_{j=1}^{N} \alpha_j\right|^{1/N}$$

We obtained in this manner $\bar{\delta} = 16 \pm 1$ and $\bar{\alpha} = 1.7 \pm 0.2$, which are within the statistical errors of the numbers obtained in ref. [3]. Note that the accuracy of the calculation is limited only by the size of our ensemble (10 000) and machine precision. Because the approximants required to ensure universality are very long periodic orbits, these computations are expensive; more accurate results would require more computer time than we currently have available.

It turns out that [‡1] our exponents $\bar{\delta}$ and $\bar{\alpha}$ for the critical map are approximately related to the scaling exponents δ and α of Jensen et al. [4]. Suppose that we iterate the transformation given by eq. (3) n times. For large n, it follows from the definition of $\bar{\delta}$ that $\Delta\omega_n \approx (\bar{\delta})^{-n}$, where $\Delta\omega_n = |\omega_{n+1} - \omega_n|$. Letting q_n be the denominator of the nth approximant of ρ we can express $\Delta\omega_n$ in terms of q_n by using the number theory result [9], $n \approx c \log q_n$ where $c = 12 \log(2)/\pi^2$. The resulting expression for $\Delta\omega_n$ is independent of the numerator of the nth approximant of ρ. Jensen et al. have found that $\Delta\omega(q) \approx q^{-\delta}$, where $\Delta\omega(q)$ is the mode-locked window width averaged over all winding numbers corresponding to a q cycle. If we make the ansatz that $\Delta\omega_n = \Delta\omega(q)$, we find that $\delta = c \log \bar{\delta}$. Similar arguments give $\alpha = c \log \bar{\alpha}$. Evaluation of the left-hand sides of these expressions give numbers in good agreement with the measured values of α and δ found in ref. [4].

The calculations described above were repeated using the map

$$x_{i+1} = x_i - \tfrac{1}{2}k\pi \sin(2\pi x_i) + a \sin^3(2\pi x_i) + \omega\,, \qquad (4)$$

with $a = -0.8$. The picture of the attractor and the critical exponents were the same for both maps, indicating universality.

The analysis we have given here is by no means the final answer to this problem. One would like to understand more about the structure of the attractor, if possible expressed in analytical form. Thus far this has

[‡1] We would like to thank Tomas Bohr for bringing this to our attention.

remained elusive. Other approaches to this problem, using a Farey tree expansion [6–8] may also shed some light on this. Note that the approach of Farmer and Satija [3] has been modified to employ a Farey tree rather than a diophantine expansion [10].

The Monte Carlo approach described here provides a simple technique to study universal properties in the general case where the renormalization transformation is chaotic. It is remarkable that the infinite-dimensional function space converges to an attractor of dimension less than two. As far as we know this is the first example in which it has been explicitly demonstrated that the renormalization transformation has a strange attractor. All the universal features of the quasiperiodic transition are contained in this attractor. In contrast to previous treatments in which only fixed points were studied, this attractor describes universal behavior of the *typical* quasiperiodic transition, and is valid for a set of winding numbers of measure one. Several important problems remain to be solved, however: First, more detailed information is needed about the structure of the universal attractor. Second, how do the results of our study relate to the results of other renormalization schemes that have recently been proposed [6–8]? We hope that our work will help stimulate research to answer these questions, and provide useful guidance in attempts to formulate this problem analytically.

This work was partially supported by the Air Force Office of Scientific Research under AFOSR grant ISSA-84-00017.

References

[1] S.J. Shenker, Physica 5D (1982) 405;
M.J. Feigenbaum, L.P. Kadanoff and S.J. Shenker, Physica 5D (1982) 370.
[2] D. Rand, S. Ostlund, J. Sethna and E.D. Siggia, Phys. Rev. Lett. 49 (1982) 132;
S. Ostlund, D. Rand, J. Sethna and E. Siggia, Physica 8D (1983) 303.
[3] J.D. Farmer and I.I. Satija, Phys. Rev. A31 (1985) 3520.
[4] M.H. Jensen, P. Bak and T. Bohr, Phys. Rev. A30 (1984) 1960.
[5] J.M. Greene, J. Math. Phys. 20 (1979) 1183.
[6] P. Cvitanovic, B. Shraiman and B. Soderberg, Scaling laws for mode lockings in circle maps, Nordita preprint 85/21 (1985).
[7] M.J. Feigenbaum, The renormalization of the Farey map and universal mode locking, Cornell University preprint.
[8] S. Ostlund and S. Kim, Universal scaling in circle maps, University of Pennsylvania preprint.
[9] I.P. Cornfeld, S.V. Fomin and Ya.G. Sinai, Ergodic theory (Springer, Berlin, 1982).
[10] P. Cvitanovic, M.H. Jensen, L.P. Kadanoff and I. Procaccia, Phys. Rev. Lett. 55 (1985) 343.