where $\vec{P}$ is the polarization density. When $\vec{P}(\vec{r}')$ is expanded to first order in $\vec{\xi}$, (6) is readily evaluated by first doing the angular integration, with the result

$$\Delta f_i^{(M)} = \nabla_j \left[\frac{(\epsilon-\epsilon_0)^2}{\epsilon_0}\left(-\tfrac{1}{5}E_iE_j + \tfrac{1}{15}\vec{E}^2\delta_{ij}\right)\right] \qquad (7)$$

which is independent of both β and the details of Φ^s.

When (2) and (7) are combined, the E_iE_j terms exactly cancel, and the sum precisely equals $\vec{f}^{(H)}$ if the Clausius-Mossotti relation is used to evaluate $\rho\,\partial\epsilon/\partial\rho$. The microscopic derivation is significant because it shows that $\vec{f}^{(H)}$ is not entirely an electrical force. Indeed, (1) should be interpreted not as the balance between a mechanical force $-\nabla\pi_0$ and an electrical force $\vec{f}^{(H)}$,[5] but rather as the balance between a mechanical force $-\nabla\pi_0 + \Delta\vec{f}^{(M)}$ and an electrical force $\vec{f}^{(E)}$. This becomes crucial when the theory is extended to time-dependent situations: $\Delta\vec{f}^{(M)}$ depends on a change of $\rho^{(2)}$ induced by the field and therefore does not assume the form given in (7) until a time on the order of the relaxation time T_c of the two-particle density, say 10^{-12} s for a liquid.

We are thus able to distinguish a number of time-dependent cases. (i) For quasistatic situations, the force density is obtained by adding the magnetic term $(\epsilon-\epsilon_0)\partial(\vec{E}\times\vec{B})/\partial t$ to $\vec{f}^{(H)}$. (ii) At higher frequencies this remains valid if all quantities are regarded as averages over several cycles. (iii) However, if the electromagnetic field is a pulse shorter than T_c—a situation not encountered experimentally up to now—$\Delta\vec{f}^{(M)}$ is not present and the force density reduces to that of Peierls.[8] Details of the time-dependent theory and a discussion on the momentum of light will be given elsewhere.[15]

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[1]F. N. H. Robinson, Phys. Rep. 16C, 313 (1975).
[2]I. Brevik, Phys. Rep. 52C, 133 (1979).
[3]A. Ashkin and J. M. Dziedzic, Phys. Rev. Lett. 30, 139 (1973); R. V. Jones and B. Leslie, Proc. Roy. Soc. London, Ser. A 360, 347 (1978).
[4]L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media* (Pergamon, New York, 1960), pp. 64–69.
[5]W. K. H. Panofsky and M. Phillips, *Classical Electricity and Magnetism* (Addison-Wesley, Reading, Mass., 1962), pp. 107–111.
[6]P. Penfield and H. A. Haus, *Electrodynamics of Moving Media* (MIT, Cambridge, Mass., 1967).
[7]J. P. Gordon, Phys. Rev. A 8, 14 (1973).
[8]R. Peierls, Proc. Roy. Soc. London, Ser. A 347, 475 (1976), and 355, 141 (1977).
[9]H. K. Wong and K. Young, Am. J. Phys. 45, 195 (1977).
[10]S. S. Hakim, Proc. Inst. Electr. Eng., Part C 109, 158 (1962).
[11]H. von Helmholtz, Wied. Ann. 13, 798 (1882).
[12]P. Mazur and S. R. de Groot, Physica (Utrecht) 22, 657 (1965); A. N. Kaufman, Phys. Fluids 8, 935 (1965).
[13]A. Einstein and J. Laub, Ann. Phys. 26, 541 (1908); H. Minkowski, Nachr. Ges. Wiss. Gottingen, 53 (1980); G. H. Livens, Philos. Mag. 32, 162 (1916).
[14]S. S. Hakim and J. B. Higham, Proc. Roy. Soc. 80, 190 (1962).
[15]H. M. Lai, W. M. Suen, and K. Young, to be published.

Spectral Broadening of Period-Doubling Bifurcation Sequences

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A perturbation calculation shows that the power spectrum of strange attractor near the accumulation parameter of a period-doubling bifurcation sequence consists of peaks broadened by a phase modulation, with broad skirts created by an amplitude modulation. Moving toward the accumulation parameter, at each bifurcation the total noise power decreases by a factor of 10.48, the average peak width decreases by a factor of 20.96, and the spectral bandwidth of the skirts decreases by a factor of 2.

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This paper discusses properties of the power spectrum of a continuous dynamical system in the chaotic regime of period-doubling sequences. The universal properties of power spectra on the periodic side of doubling sequences were originally discussed by Feigenbaum,[1] and his predictions are in qualitative agreement with convection experiments by Libchaber and Maurer,[2] and Gollub, Benson, and Steinman.[3] Since there is numerical evidence that mathematical models with period-doubling sequences contain strange attractors, this agreement supports the hope that chaotic fluid flow can be modeled by strange attractors. The results presented here provide a more severe test of this theory. They support previous[4,5] and concurrent[6] work, and, in addition, treat the general case of dynamical systems that are not periodically driven.

A dynamical system with a period-doubling sequence that accumulates at parameter r_c has on one side a sequence of limit cycles whose period repeatedly doubles as r_c is approached. On the other side of r_c, numerical evidence[7-9] indicates that there is a sequence of strange attractors as shown in Fig. 1. To a good approximation, a strange attractor near r_c is a thin two-dimensional ribbon that makes 2^n loops and then closes onto itself. Aspects of this behavior can be summarized with use of a return map, constructed as follows: The intersection of the attractor and a transverse surface is approximately a curve. When this curve is parametrized by a variable y, successive crossings at times t_i yield a sequence y_i given by a recursion relation $y_{i+1}=F(y_i)$, where F is a continuous function (see Shaw[10]).

On the chaotic side, near r_c the probability density of y_i is nonzero on 2^n bands, corresponding to the 2^n loops of the continuous attractor. Motion between bands is periodic with period 2^n, but motion within each band is chaotic. This chaotic motion, which introduces broad components into a power spectrum, can be thought of as an amplitude modulation of an otherwise periodic orbit.

For a limit cycle, for example, the sequence of return times $T_i=t_{i+1}-t_i$ is constant. The return times T_i are also constant for a strange attractor of a periodically driven system, as long as the surface of section used to construct the return map is taken at a constant phase of the driving force. The power spectrum in this case contains δ-function peaks superimposed on the broad background created by the amplitude modulation.

For the more general case, the return times T_i are *not* periodic. Nevertheless, numerical evidence indicates that the chaotic sequence T_i can be approximated as a continuous function of y_i, i.e., $T_i=T(y_i)$. Thus, orbits can gain or lose phase due to the chaotic behavior of $T(y_i)$. Letting $T_0=\langle T_i\rangle$ (time average), and $\omega_0=2\pi f_0=2\pi/T_0$, the net phase fluctuation in completing a cycle is $\delta\theta_i=\omega_0(T_i-T_0)$. The chaotic return times effectively create a random "phase modulation" that broadens the peaks of the power spectrum.

When the central-limit theorem holds for $\delta\theta_i$, it ensures that the cumulative phase drift $\theta_k=\theta(t_k)=\sum_{i=1}^{k}\delta\theta_i$ has a Gaussian probability density for large k. Ratner[11] has shown that the central-limit theorem holds for dynamical systems that satisfy Axiom A.[12] Unfortunately, there are no known dynamical systems that satisfy Axiom A and also have a period-doubling sequence. Fortunately, behavior qualitatively similar to that in which we

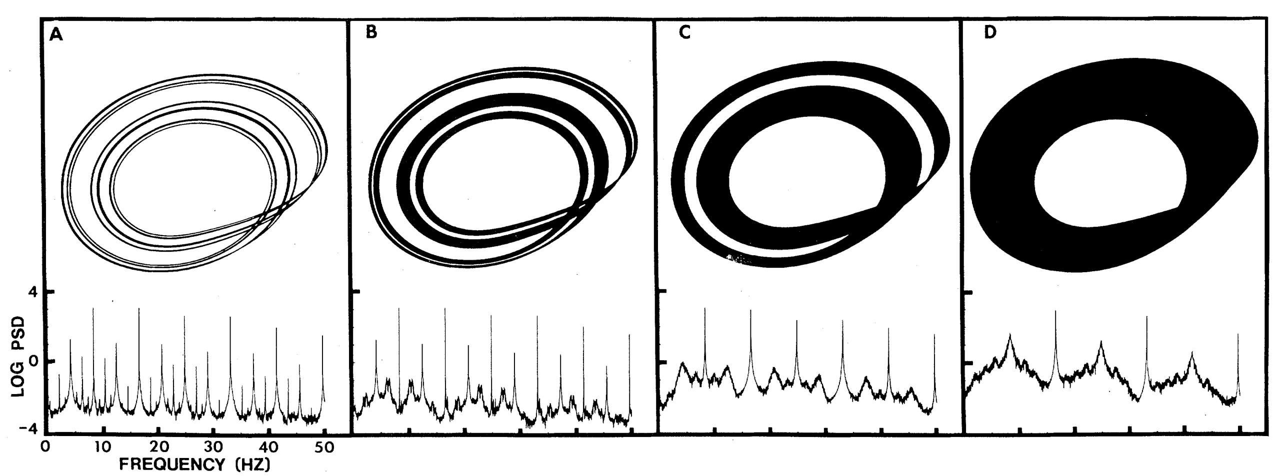

FIG. 1. Four simulations of strange attractors of the Rössler dynamical system, taken from Ref. 9. Case (a) is closest to r_c, and is a period-8 attractor. A power spectrum is shown below each frame.

are interested can be simulated by "arbitrarily" choosing a return map F with a period-doubling sequence, "arbitrarily" choosing a continuous time transformation T, and using this pair to generate sequences $\delta\theta_i$. In every case studied, the coarse-grained probability density $\chi(\theta_k)$ approached a Gaussian. (See Fig. 2.)

Naturally, as time increases the spread in the cumulative phase fluctuations θ_k gets larger. It can be shown that the variance σ^2 of $\chi(\theta_k)$ asymptotically grows linearly in time at a rate c. (This proof assumes that the sum of the autocorrelation function of T_i is finite.) For a limit cycle, or a periodically driven system, $\delta\theta_i = 0$, which implies that $c = 0$. In general, however, $c \neq 0$.

We are now ready to compute the form of the power spectrum. As a first approximation, the attractor is a limit cycle $p(\omega_0 t)$, with period $2^n(2\pi)$. To take the chaotic motion into account, write the transverse displacement from the limit cycle p as $w(\omega_0 t)R(\omega_0 t)$. $w(\omega_0 t)$ is the width of the attractor at phase $\omega_0 t$, and is periodic with period $2^n(2\pi)$. Thus, all of the chaotic behavior of the amplitude is contained in R. To take into account the chaotic phase drifting, write the phase at time t as $\varphi(t) = \omega_0 t + \theta(t)$. A trajectory on the attractor can be written as

$$x(t) = p(\varphi(t)) + w(\varphi(t))R(\varphi(t)). \quad (1)$$

$p(\varphi(t))$ is constructed so that $\langle x(t)\rangle = p(\omega_0 t)$, and R is constructed so that $\langle R\rangle = 0$.

A complication in the application of these ideas is that experiments are normally conducted with use of a projection onto a single coordinate. All of the following remarks remain true, however, if x, R, p, w, and y_i are consistently considered to be the projected values.

The autocorrelation of x can be computed from Eq. (1) by assuming that x is uncorrelated with p and w (see Thomae and Grossman[4]):

$$Q_x(t) = Q_p(t) + Q_w(t)Q_R(t). \quad (2)$$

Q_x, Q_p, Q_w, and Q_R are the autocorrelation functions of x, p, w, and R, respectively. In the absence of phase fluctuations, $c = 0$, $\varphi(t) = \omega_0 t$, and therefore $Q_p(t)$ and $Q_w(t)$ are periodic. Including phase fluctuations has the effect of multiplying Q_p and Q_w by a damping factor $e^{-ct/2}$. [To do this calculation it is necessary to assume that $\theta(t)$ is ergodic with a Gaussian probability density, and convert time averages to ensemble averages.] Letting P_k and W_k be the complex Fourier coefficients of p and w, and $f_k = (k/2^n)f_0$, Fourier transforming Eq. (2) gives the power spectrum of x,

$$S_x(f) = 2\sum_{k=1}^{\infty}\left[\,|P_k|^2 L_c(f-f_k) + |W_k|^2 S_R(f-f_k)\right]. \quad (3)$$

S_R is the power spectrum of R, and $L_c(f-f_k)$ is a Lorentzian peak of half power width $c/4\pi$ centered at f_k, i.e.,

$$L_c(f-f_k) = 2c/\{c^2 + [4\pi(f-f_k)]^2\}. \quad (4)$$

The effect of phase modulations has been neglected in the second term of Eq. (3), since for small values of c this is a second-order effect. In the first term, however, the phase modulations are responsible for the broadening of the δ-function peaks into Lorentzians. We will refer to the

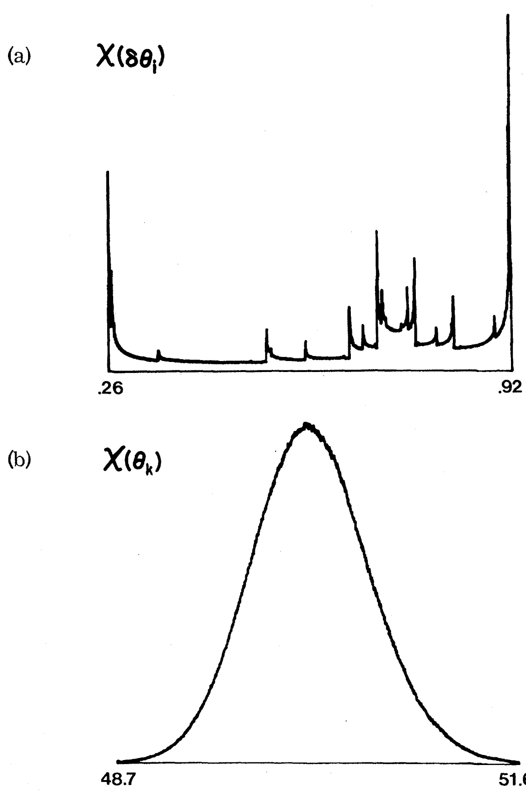

Fig. 2. A coarse-grained probability density of $\delta\theta_i = T(y_i)$, obtained by iterating, by 10^6 times, the one-dimensional map $y_{i+1} = 3.7y_i(1-y_i)$ and sorting the result into 1000 bins in order to estimate the frequency of occurrence over each bin. For this case $T(y) = y$. (b) Similar to (a), except the probability density is constructed for $\theta_k = \sum_{i=1}^{k}\delta\theta_i$, with $k = 75$. Several different choices of smooth time transformations T all show $\chi(\theta_{75})$ approximately a Gaussian.

terms $|W_k|^2 S_R(f-f_k)$ as "skirts" because they are convolved about each peak. Taken together, they form a broad background caused by the amplitude modulation.

As a parameter r is varied, the power spectrum changes in a manner that becomes universal[1] as r approaches r_c. Let r_n be the parameter value where the number of distinct bands in the return map goes from 2^n to 2^{n-1}. At any given parameter value r_n, the width $w_i(r_n)$ of each band is not constant, and varies considerably in completing a cycle. Nevertheless, our numerical investigations show that

$$\lim_{n\to\infty} [\langle w_i^2(r_n)\rangle / \langle w_i^2(r_{n+1})\rangle] \to \gamma, \qquad (5)$$

where $\gamma \cong 10.48$ is a universal number (see also Ref. 6). Parseval's theorem implies that the total noise power $\sum_{k=1}^{\infty} W_k^2$ also decreases by a factor of γ at each bifurcation. In addition, the ratio of the square of the separation of the adjacent bands at r_n to that at r_{n+1} is given by γ. This implies that the total power added to the periodic part of the spectrum in going from r_n to r_{n+1} is a factor of γ smaller than that added in going from r_{n-1} to r_n. A more detailed prediction of the behavior of the Fourier coefficients P_k (which behave just as they do on the periodic side of the bifurcation sequence) has been made by Feigenbaum,[1] and, with somewhat different results, by Nauenberg and Rudnick.[13] At $r=r_n$, if the 2^n iterate of F is restricted to a given band and rescaled appropriately, a universal function is approached. In passing to $r=r_{n+1}$ the 2^{n+1} iterate must be used; the number of iterations needed to construct a universal function consequently doubles. As a result, in passing from r_n to r_{n+1}, the frequency of S_R must be rescaled by a factor of 2, i.e., $2S_R(r_{n+1}, 2f) = S_R(r_n, f)$. (The factor of 2 in amplitude is necessary to maintain the integral of S_R constant.) Thus, the characteristic frequency of S_R at $r=r_{n+1}$ is half that at $r=r_n$.

The half power width of the Lorentzian peaks in the spectrum depends on c, the rate of growth of the variance of the cumulative phase fluctuations. At each bifurcation, the decrease by γ in the mean-square width of the bands causes a corresponding decrease in the mean-square value of the phase fluctuations $\delta\theta_i$. In addition, twice as many iterations are needed to complete a cycle and return to a universal function; the rate c must decrease altogether by a factor of $2\gamma \cong 20.96$ at each bifurcation. In passing through successive bifurcations c decreases rapidly, justifying the assumptions used to compute Eq. (4). After only a few bifurcations the peaks become experimentally indistinguishable from δ functions. This explains the sharpness of the peaks seen by Gollub, Benson, and Steinman.[3] It does not explain, however, why these sharp peaks frequently persist long after all the bands merge, far away from r_c.[8,14]

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[1]M. J. Feigenbaum, J. Stat. Phys. 19, 25 (1978), and 21, 669 (1979).

[2]A. Libchaber and J. Maurer, J. Phys. (Paris), Colloq., 41, C3-51 (1980).

[3]J. Gollub, S. Benson, and J. Steinman, Ann. N.Y. Acad. Sci. 357, 22 (1980).

[4]S. Thomae and S. Grossman, "Correlations and Spectra of Periodic Chaos Generated by the Logistic Parabola" (to be published), and "A Scaling Property in Critical Spectra of Discrete Systems" (to be published).

[5]A. Wolf and J. Swift, "Universal Power Spectra of Strange Attractors" (to be published).

[6]B. Huberman and A. Zisook, Phys. Rev. Lett. 46, 626 (1981).

[7]S. Grossman and S. Thomae, Z. Naturforsch. 32A, 1353 (1977).

[8]E. N. Lorenz, Ann. N.Y. Acad. Sci. 357, 282 (1980).

[9]J. Crutchfield, D. Farmer, N. Packard, R. Shaw, G. Jones, and R. Donnelly, Phys. Lett. 76A, 1 (1980).

[10]R. Shaw, Z. Naturforsch. 36A, 80 (1981).

[11]M. Ratner, Isr. J. Math. 16, 181 (1973).

[12]S. Smale, Bull. Amer. Math. Soc. 13, 747 (1967).

[13]M. Nauenberg and J. Rudnick, "Universality and the Power Spectrum at the Onset of Chaos" (to be published).

[14]D. Farmer, J. Crutchfield, H. Froehling, N. Packard, and R. Shaw, Ann. N.Y. Acad. Sci. 357, 453 (1980).