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Generalized Lyapunov exponents corresponding to higher derivatives

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We generalize the notion of Lyapunov exponents to higher order derivatives. For fixed points and periodic orbits we derive a relationship for the higher order Lyapunov spectrum in terms of the usual first order Lyapunov spectrum. Based on numerical experiments as well as general arguments, we conjecture that this relationship also holds for chaotic orbits. This work is relevant to a priori error estimates for time series forecasting.

1. Introduction

The Lyapunov characteristic exponents [6] are one of the most useful tools for characterizing dynamical systems. They characterize the average local stability properties of a dynamical system, and to first order describe the rate at which small volumes expand or contract in different directions. They can be defined intuitively by considering an n-dimension dynamical system $x_t = f^t(x)$, and an infinitesimally small ball of radius $r(0)$, centered at position x. Since the ball is infinitesimal, under the action of the dynamical system its shape is affected only by the linear part of f^t. At time t it evolves into an ellipsoid. Calling the principal axes of the ellipsoid $r_i(t)$, the spectrum of Lyapunov exponents is

$$\lambda_i(x) = \lim_{t\to\infty} \lim_{r(0)\to 0} \frac{1}{t} \log \frac{r_i(t)}{r(0)}. \tag{1}$$

The "small ball" discussed above can be thought of as a convenient model of measurement error. In any real experiment, however, measurement errors are *finite*. This means that the geometric form that the initial ball evolves into is only an ellipsoid to first order; in general it is distorted by higher order nonlinearities. When the dynamics is locally unstable the nonlinearities become more severe as the ellipsoid expands. By expanding in a Taylor series the nonlinear distortions can be grouped according to their quadratic, cubic, etc. parts. The higher order Lyapunov exponents that we develop here characterize the average rate of growth of these nonlinear distortions at each order. Just as the first order Lyapunov exponents describe the average rate of growth (or decrease) of the first derivative, the higher order Lyapunov exponents describe the average rate of growth (or decrease) of higher derivatives.

Farmer and Sidorowich [3,4] originally introduced higher order Lyapunov exponents in one dimension, and conjectured an approximate relationship for the higher order exponents in terms

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of the first order exponents. In this paper we prove that for periodic orbits their conjecture is actually an equality. Furthermore, we give a precise definition of the higher order exponents in more than one dimension, and prove that for fixed points and limit cycles the higher order exponents can be expressed exactly in terms of the first order exponents. We present numerical evidence supporting our conjecture that this relationships hold for chaotic orbits as well.

Our interest in higher order Lyapunov exponents is motivated by practical problems in time series modeling. The problem is to form non-linear models directly from time series data, and use these models to make predictions about the future behavior of the data. For chaotic systems the prediction errors generally increase as we extrapolate further into the future. At what rate? For a given approximation method, what does this rate depend on?

Suppose that the "true" dynamics $f'(x)$ is approximated by $\hat{f}'(x)$. The approximation error at x is just $\|f'(x) - \hat{f}'(x)\|$. An approximation method is defined to be of *order q* if the error depends on the qth derivative $d_x^q f'$. For example, consider approximation in terms of piece-wise linear elements (linear interpolation). Expanding f' in a Taylor series shows that $\|f'(x) - \hat{f}'(x)\|$ is proportional to the second derivative $d_x^2 f'$. This means that the average rate of growth of the error is proportional to the average rate of growth of $d_x^2 f'$. This is described by the second order Lyapunov exponents.

This paper is organized as follows: In section 2 we derive our basic results for fixed points and limit cycles in one dimension, where the mathematics is simpler. In section 3, we extend these results to the multidimensional case. In section 4 we argue that these results should generically hold for chaotic orbits as well, and in section 5 we present supporting numerical evidence.

For convenience all of our formal results concern the second order Lyapunov exponents. Higher order exponents are a straightforward (but tedious) extension.

The reader who only wishes to see a precise statement of the definition of higher order exponents should see section 3.2. For a precise statement of our mains results see eqs. (49)–(51).

2. One dimensional dynamics

To illustrate the basic ideas with a minimum of mathematical formalism, we begin by considering one dimensional dynamical systems. The extension to higher dimensional systems (which brings in some complications) is treated in the following section.

To define the qth order Lyapunov exponent in one dimension, consider a one dimensional map, $x_{t+1} = f(x_t)$. Assume that f is q times differentiable. For convenience we define $x = x_0$, and use the notation $x_t^{(q)} = d_x^{(q)} f^t = d^q f^t/dx^q(x)$ for the qth derivative of the tth iterate of f, or alternatively for brevity $x_t' = x_t^{(1)}$, $x_t'' = x_t^{(2)}$, etc. The qth order Lyapunov exponent is

$$\lambda^{(q)}(x_0) = \lim_{t\to\infty} \frac{1}{t} \log|x_t^{(q)}|. \tag{2}$$

For most of the paper we will restrict our attention to the second order Lyapunov exponent $\lambda^{(2)}$.

2.1. Decomposition of the second derivative of f^t

We begin by recursively differentiating:

$$x_{t+1} = f(x_t),$$

$$x_{t+1}' = f'(x_t)x_t',$$

$$x_{t+1}'' = f'(x_t)x_t'' + f''(x_t)(x_t')^2. \tag{3}$$

If we assume that f is ergodic with a natural measure, because the recursion relation for x_t' gives x_t' as a simple product, the first order Lyapunov exponent can be written as an ensemble average $\lambda^{(1)} = \langle \log|f'(x)|\rangle_x$. This does not

hold for general $\lambda^{(q)}$ because the higher derivatives involve sums as well as products.

The recursion relation for x_t'' can be explicitly resolved as

$$x_t'' = \sum_{m=0}^{t-1} \left(\prod_{n=m+1}^{t-1} f'(x_n) \right) \cdot f''(x_m) \cdot \left(\prod_{n=0}^{m-1} f'(x_n) \right)^2 .$$

(4)

2.2. Fixed points

For a fixed point x ($x = x_t, \forall t$) it is straightforward to relate the second order Lyapunov exponent to the first. Eq. (4) is easily rewritten:

$$x_{t+1}'' = f''(x) \cdot \sum_{m=0}^{t} f'(x)^{t+m} .$$

(5)

The leading behavior of this expression depends on the stability of the fixed point. We restrict ourselves to the case $f''(x) \neq 0$. (When $f''(x) = 0$, $\lambda^{(2)}(x) = -\infty$.)

2.2.1. Stable fixed point ($|f'(x)| < 1$)

$$x_{t+1}'' = f'(x)^t \cdot f''(x) \cdot \sum_{m=0}^{t} f'(x)^m$$

$$= f'(x)^t \cdot f''(x) \cdot \frac{1 - f'(x)^{t+1}}{1 - f'(x)}$$

$$= f'(x)^t \cdot C(x, t) ,$$

(6)

where

$$C(x, t) = f''(x) \cdot \frac{1 - f'(x)^{t+1}}{1 - f'(x)}$$

is bounded and $C(x, t) \neq 0$ for all t. Combining eqs. (2) and (6),

$$\lambda^{(2)}(x) = \log|f'(x)| = \lambda^{(1)}(x) .$$

(7)

2.2.2. Unstable fixed point ($|f'(x)| > 1$)
Reordering the sum in (5) we obtain

$$x_{t+1}'' = f''(x) \cdot \sum_{m=0}^{t} f'(x)^{2t-m}$$

$$= f'(x)^{2t} \cdot f''(x) \sum_{m=0}^{t} f'(x)^{-m}$$

$$= f'(x)^{2t} \cdot D(x, t) ,$$

(8)

with

$$D(x, t) = f''(x) \cdot \frac{1 - f'(x)^{-(t+1)}}{1 - f'(x)^{-1}}$$

again bounded and nonzero.

Therefore $\lambda^{(2)}(x)$ for an unstable fixed point is

$$\lambda^{(2)}(x) = 2\log|f'(x)| = 2\lambda^{(1)}(x) .$$

(9)

2.2.3. Nonhyperbolic fixed point ($|f'(x)| = 1$)

$$x_{t+1}'' = f'(x)^t \cdot f''(x) \cdot \sum_{m=0}^{t} f'(x)^m$$

$$= f'(x)^t \cdot G(x, t) .$$

(10)

This time $G(x, t)$ is not bounded, but grows at most linearly, i.e. $|G(x, t)| \leq |f''(x)| \cdot (t + 1)$. This growth is small enough that

$$\lambda^{(2)}(x) = \log|f'(x)| = \lambda^{(1)}(x) .$$

(11)

All three cases can be summarized by

$$\lambda^{(2)}(x) = \max\{2\lambda^{(1)}(x), \lambda^{(1)}(x)\} .$$

(12)

Placing the result in this form is particularly appropriate for the extension to the multidimensional case.

2.3. Periodic orbits

Since a periodic orbit is just a fixed point of the pth iterate, we expect our results to extend to periodic orbits in the natural way, i.e., letting $\lambda^{(2)}(f^p, x)$ be the 2nd order Lyapunov exponent of the map f^p,

$$\lambda^{(2)}(f^p, x) = p\lambda^{(2)}(f, x) .$$

(13)

As we demonstrate in the appendix, for an orbit of period p

$$x''_{(k+1)p} = c(x) \cdot \sum_{j=0}^{k} (x'_p)^{k+j} , \qquad (14)$$

where

$$c(x) = \sum_{r=0}^{p-1} f''(x_r) \prod_{l=r+1}^{p-1} f'(x_l) \prod_{n=0}^{r-1} f'(x_n)^2 , \qquad (15)$$

$c(x)$ is independent of k for fixed p. It is therefore bounded for fixed p. Comparing (14) with (5), we see that this is the same form as for the fixed point, with $f''(x)$ replaced by $c(x)$. Since for a fixed point $\log|x'_p| = p\lambda^{(1)}(x)$, we can follow the same chain of reasoning used to get from (5) to (12). Thus, letting x_i represent the points on the periodic orbit, providing

$$c(x_i) \neq 0 , \quad i = 0, p-1 , \qquad (16)$$

we get the same result that

$$\lambda^{(2)}(x) = \lim_{k \to \infty} \frac{1}{kp} \log|x''_{kp}|$$

$$= \max\{2\lambda^{(1)}(x), \lambda^{(1)}(x)\} \qquad (17)$$

for any point $x = x_i$, independent of i.

3. n-dimensional dynamics

3.1. Review

3.1.1. Basic notation

In this section we review a few very basic facts of differential calculus that will be necessary in what follows, in order to establish notation. We have a dynamical system on a n-dimensional manifold M:

$$x_{t+1} = f(x_t) , \quad x_t \in M . \qquad (18)$$

The derivative $d_x f$ at a point $x \in M$ is a linear mapping from $T_x M$ to $T_{f(x)} M$,

$$d_x f: T_x M \to T_{f(x)} M . \qquad (19)$$

$T_x M$ denotes the tangent space at x, which is isomorphic to $\mathbb{R}^n$ for every x. If $v \in T_x M$, then $d_x f(v) \in T_{f(x)} M$. In a specific coordinate system the kth component of $d_x f(v)$, $(d_x f(v))_k$, is given by

$$(d_x f(v))_k = \sum_{i=1}^{n} \frac{\partial f_k}{\partial x_i} v_i . \qquad (20)$$

The second derivative $d_x^2 f$ at a point $x \in M$ can be thought as a symmetric bilinear form

$$d_x^2 f: T_x M \times T_x M \to T_{f(x)} M . \qquad (21)$$

Symmetric means that $d_x^2 f(v, w) = d_x^2 f(w, v)$ for $v, w \in T_x M$, and bilinear refers to the fact that $d_x^2 f$ is linear in both arguments separately.

In a given coordinate system $d_x^2 f(v, w)$ has the form

$$(d_x^2 f(v, w))_k = \sum_{i,j=1}^{n} \frac{\partial f_k}{\partial x_i \, \partial x_j} v_i w_j , \qquad (22)$$

where $(d_x^2 f(v, w))_k$ is the kth component of $d_x^2 f(v, w) \in T_{f(x)} M$. The extension to higher order derivatives $d_x^q f$ proceeds similarly.

In the following section it will be convenient to use the tensor product $d_x f' \otimes d_x f'$, defined as

$$d_x f' \otimes d_x f': T_x M \times T_x M \to T_{x_t} M \times T_{x_t} M , \qquad (23)$$

where $d_x f' \otimes d_x f'(v, w) = (d_x f'(v), \; d_x f'(w))$ with $v, w \in T_x M$. Note that on the direct product $T_x M \times T_x M$ with elements (v, w) $d_x f' \otimes d_x f'$ is a bilinear form while on the tensor product $T_x M \otimes T_x M$ with corresponding elements $v \otimes w$ $d_x f' \otimes d_x f'$ can be considered as a linear mapping.

3.1.2. First order Lyapunov exponents

For a variation in a given direction described

by $u \in T_x M$, the associated (first order) Lyapunov exponent is

$$\lambda^{(1)}(x, u) = \lim_{t \to \infty} \frac{1}{t} \log \| d_x f^t(u) \| . \tag{24}$$

Oseledec's theorem [6] allows us to group the Lyapunov exponents asociated with different tangent vectors, so that rather than associating an exponent with each u, we can associate a spectrum of exponents with each point x, and summarize in this way the behavior of all u. According to Oseledec's theorem, the matrix

$$\Lambda_x = \lim_{t \to \infty} ((d_x f^t)^{tr} d_x f^t)^{1/2t} \tag{25}$$

exists and has positive eigenvalues $e^{\lambda_1^{(1)}(x)} \geq \cdots \geq e^{\lambda_n^{(1)}(x)}$, where $\lambda_i^{(1)}(x)$ are the Lyapunov exponents of x, ordered according to size, i.e. $\lambda_1^{(1)}(x) \geq \cdots \geq \lambda_n^{(1)}(x)$. Suppose we identify degenerate Lyapunov exponents, and let m^i be the multiplicity of $\lambda_i^{(1)}(x)$. Furthermore, let $V_x^i \subset \mathbb{R}^n$ the eigenspaces of $\lambda_i^{(1)}(x)$, i.e. dim $V_x^i = m^i$ and $\Lambda_x v_i = e^{\lambda_i^{(1)}(x)} v_i$, $\forall v_i \in V_x^i$. The V_x^i are orthogonal to each other and span $\mathbb{R}^n$, i.e. $\mathbb{R}^n = V_x^1 \oplus \cdots \oplus V_x^{n'}$. ($\oplus$ denotes the direct sum of vector spaces). Letting n' denote the number of different Lyapunov exponents, the filtration E_x^i associated by the dynamics is given by

$$E_x^i = V_x^i \oplus \cdots \oplus V_x^{n'} . \tag{26}$$

Using the "set subtraction" symbol $A \backslash B = \{x: x \in A \text{ but } x \notin B\}$, we can alternatively define the spectrum of Lyapunov exponents as

$$\lambda_i^{(1)}(x) = \lim_{t \to \infty} \frac{1}{t} \log \| d_x f^t u \| \quad \text{if } u \in E_x^i \backslash E_x^{i+1} . \tag{27}$$

In other words, the ith Lyapunov exponent describes the rate of separation for variations in directions excluded from those described by larger exponents.

3.2. Definition of higher order Lyapunov exponents

We will define the second order Lyapunov exponent associated with a given point x and a pair of tangent vectors u and v as

$$\lambda^{(2)}(x, u, v) = \lim_{t \to \infty} \frac{1}{t} \log \| d_x^2 f^t(u, v) \| . \tag{28}$$

Note that since $d_x^2 f^t$ is a symmetric bilinear form on $T_x M \times T_x M$ to $T_{x_t} M$ it can be considered a linear mapping from $T_x M \otimes_s T_x M$, where $\otimes_s$ is the symmetric tensor product. Note that because the dimension of $T_x M \otimes_s T_x M$ is $\frac{1}{2} n(n + 1)$, for $n > 1$, $d_x^2 f^t$ has a nonzero kernel. In this case, $\lambda^{(2)}(x, u, v) = -\infty$.

Letting $w_l \in E_x^l \backslash E_x^{l+1}$ (as defined above), we can summarize the possible growth rates of the pairs (u, v) by defining the *spectrum of* second order exponents associated with x as

$$\lambda_{ij}^{(2)}(x) = \lambda^{(2)}(x, w_i, w_j) , \quad i, j = 1, \dots, n' . \tag{29}$$

The definition of $\lambda_{i_1 \cdots i_q}^{(q)}(x)$, for $q > 2$, proceeds analogously.

3.3. Decomposition of the second derivative of f^t

For convenience we take $x_0 = x$ so that $x_t = f^t(x)$. In n dimensions the recursion relations of eq. (3) become

$$x_{t+1} = f(x_t) ,$$

$$d_x f^{t+1} = d_{x_t} f \cdot d_x f^t ,$$

$$d_x^2 f^{t+1} = d_{x_t}^2 f \cdot (d_x f^t \otimes d_x f^t) + d_{x_t} f \cdot d_x^2 f^t . \tag{30}$$

In one dimension the ordering of the recursion relations is not important, because the derivatives are simply real numbers and they commute. In more than one dimension this is not true.

The resolved recursion relation analogous to eq. (4) is

$$d_x^2 f^t = \sum_{m=0}^{t-1} (d_{x_{t-1}} f \cdots d_{x_{m+1}} f) \cdot d_{x_m}^2 f$$
$$\cdot (d_{x_{m-1}} f \cdots d_x f) \otimes (d_{x_{m-1}} f \cdots d_x f) .$$
(31)

Using the chain rule for the first derivative, i.e.

$$d_x f^k = d_{x_{k-1}} f \cdots d_{x_1} f \cdot d_x f ,$$
(32)

eq. (31) can be rewritten as

$$d_x^2 f^t = \sum_{m=0}^{t-1} d_{x_{m+1}} f^{t-1-m} \cdot d_{x_m}^2 f \cdot (d_x f^m \otimes d_x f^m) .$$
(33)

3.4. Fixed points

3.4.1. Review of first order exponents for fixed points

Let x be a fixed point $f(x) = x$. Furthermore, for simplicity assume the linearization $d_x f$ of the map at the fixed point is semisimple, i.e. complex diagonalizable [5]. With this assumption there exist n' distinct eigenvalues $\mu_i \in \mathbb{C}$, $i = 1$, $\ldots$, n', ordered by size of the absolute value $|\mu_1| > \cdots > |\mu_{n'}| > 0$ with the property that if $\mu_i \in \mathbb{R}$ then there exists an eigenspace $U_x^i \subset \mathbb{R}^n$ such that $d_x f u_i = \mu_i u_i$ for all $u_i \in U_x^i$. Otherwise if $\mu_i \in \mathbb{C}\backslash\mathbb{R}$, then the complex conjugate $\bar{\mu}_i$ is also an eigenvalue and there exists a two dimensional real subspace U_x^i such that $d_x f U_x^i = U_x^i$ such that $d_x f$ restricted to U_x^i is just a rotation and stretching by $|\mu_i|$. For this case it can be shown that the average stretching rate of a vector $u_i \in U_x^i$ is given by $\log|\mu_i|$ and that the Lyapunov exponents of first order are

$$\lambda_i^{(1)}(x) = \log|\mu_i| , \quad i = 1, \ldots, n' .$$
(34)

Furthermore, the tangent space can be decomposed into the direct sum of the eigenspaces, i.e.

$$T_x M = U_x^1 \oplus \cdots \oplus U_x^{n'}$$
(35)

and each vector $u \in T_x M$ can be written uniquely as $u = \Sigma_{i=1}^{n'} c_i u_i$, $u_i \in U_x^i$ with $\|u_i\| = 1$. Generally $\{u_i\}$ is not orthonormal, and U_x^i and V_x^i are not equivalent.

In the following for convenience we assume that the eigenvalues μ_i are real.

$$d_x f u_i = \mu_i u_i , \quad i = 1, \ldots, n' .$$
(36)

In a similar manner to the arguments leading (34), the argument can also be made rigorous for complex eigenvalues (for the appropriate notation see [2]).

In terms of these eigenspaces the associated splitting of the tangent space is given for a fixed point by

$$E_x^i = U_x^i \oplus U_x^{i+1} \oplus \cdots \oplus U_x^{n'} .$$
(37)

3.4.2. Computation of second order exponents

For a fixed point we can compute the second order Lyapunov exponents from the definitions (28) and (29). We restrict ourselves first on the case where $u_i \in U_x^i$, i.e. we evaluate (29) first on eigendirections of the fixed point x which are special vectors in $E_x^i \backslash E_x^{i+1}$. We must first compute $d_x^2 f^t$.

For a fixed point $d_x f^t = (d_x f)^t$. Making use of this, and applying equation (33) to the pair of tangent vectors (u_i, u_j) yields

$$d_x^2 f^{t+1}(u_i, u_j)$$
$$= \sum_{m=0}^{t} (d_x f)^{t-m} d_x^2 f((d_x f)^m u_i, (d_x f)^m u_j) .$$
(38)

Using the bilinearity of $d_x^2 f$ and (36),

$$d_x^2 f^{t+1}(u_i, u_j) = \sum_{m=0}^{t} \mu_i^m \mu_j^m (d_x f)^{t-m} d_x^2 f(u_i, u_j) .$$
(39)

Although U_x^i and U_x^j are invariant under $d_x f$, in

general $U_x^i \times U_x^j$ is not mapped into U_x^i or U_x^j by $d_x^2 f$. Let l' denote the most unstable direction where (u_i, u_j) has a component after it is mapped by $d_x^2 f$. (Naturally, if $\lambda_{l'} < 0$, then by "most unstable" we mean "least stable"). By definition, then, $d_x^2 f(u_i, u_j)$ is in $E_x^{l'} \backslash E_x^{l'+1}$, and can be written

$$d_x^2 f(u_i, u_j) = \sum_{k=l'}^{n'} c_k u_k \quad \text{with } c_{l'} \neq 0 . \tag{40}$$

Using (40), the linearity of $d_x f$ and (36) we obtain

$$d_x^2 f^{t+1}(u_i, u_j) = \sum_{m=0}^{t} \mu_i^m \mu_j^m \sum_{k=l'}^{n'} \mu_k^{t-m} c_k u_k . \tag{41}$$

This can finally be written as

$$d_x^2 f^{t+1}(u_i, u_j) = \sum_{k=l'}^{n'} c_k \kappa_k(t) \, u_k , \tag{42}$$

with

$$\kappa_k(t) = \sum_{m=0}^{t} \mu_i^m \mu_j^m \mu_k^{t-m} . \tag{43}$$

Now we investigate the scaling of $\kappa_k(t)$, $k \in \{l', \dots, n'\}$ distinguishing three cases:

(a) k: with $|\mu_k| < |\mu_i \mu_j|$, i.e. $\lambda_i^{(1)} + \lambda_j^{(1)} > \lambda_k^{(1)}$.

$$\begin{aligned} \kappa_k(t) &= \mu_i^t \mu_j^t \cdot \sum_{m=0}^{t} \mu_i^{(m-t)} \mu_j^{(m-t)} \mu_k^{t-m} \\ &= \mu_i^t \mu_j^t \cdot \sum_{m=0}^{t} \left(\frac{\mu_k}{\mu_i \mu_j} \right)^{t-m} \\ &= \mu_i^t \mu_j^t \cdot \sum_{m=0}^{t} \left(\frac{\mu_k}{\mu_i \mu_j} \right)^{m} . \end{aligned} \tag{44}$$

Therefore $\kappa_k(t)$ can be written as $\kappa_k(t) = \mu_i^t \mu_j^t \cdot \delta_k(t)$ with $|\delta_k(t)| < C$ and $\delta_k(t) \neq 0$ for $\mu_k \neq 0$.

(b) k: with $|\mu_k| > |\mu_i \mu_j|$, i.e. $\lambda_i^{(1)} + \lambda_j^{(1)} < \lambda_k^{(1)}$.

$$\kappa_k(t) = \mu_k^t \sum_{m=0}^{t} \mu_i^m \mu_j^m \cdot \mu_k^{-m} = \mu_k^t \sum_{m=0}^{t} \left(\frac{\mu_i \mu_j}{\mu_k} \right)^{m} . \tag{45}$$

Again $\kappa_k(t) = \mu_k^t \cdot \delta_k(t)$ with $0 < |\delta_k(t)| < C$ for $\mu_k \neq 0$.

(c) k: with $|\mu_i \mu_j| = |\mu_k|$, i.e. $\lambda_i^{(1)} + \lambda_j^{(1)} = \lambda_k^{(1)}$.

In this case we obtain:

$$\kappa_k(t) = \mu_i^t \mu_j^t \cdot \delta_k(t) \tag{46}$$

with $\delta_k(t)$ linearly bounded, $|\delta_k(t)| \leq t + 1$.

We can now split the sum of equation (42) into its three possible cases:

$$\begin{aligned} d_x^2 f^{t+1}(u_i, u_j) = \mu_i^t \mu_j^t \sum_{k:|\mu_k|<|\mu_i \mu_j|} \delta_k(t) c_k u_k \\ + \sum_{k:|\mu_k|>|\mu_i \mu_j|} \mu_k^t \delta_k(t) c_k u_k \\ + \mu_i^t \mu_j^t \sum_{k:|\mu_k|=|\mu_i \mu_j|} \delta_k(t) c_k u_k . \end{aligned} \tag{47}$$

Note that $k \geq l'$. Now we use $|\mu_{l'}| \geq |\mu_k|$, for $k = l', \dots, n'$ to obtain:

$$\begin{aligned} d_x^2 f^{t+1}(u_i, u_j) &= \mu_i^t \mu_j^t \sum_{k:|\mu_k|<|\mu_i \mu_j|} \delta_k(t) \, c_k u_k \\ &+ \mu_{l'}^t \sum_{k:|\mu_k|>|\mu_i \mu_j|} \left(\frac{\mu_k}{\mu_{l'}} \right)^t \delta_k(t) \, c_k u_k \\ &+ \mu_i^t \mu_j^t \sum_{k:|\mu_k|=|\mu_i \mu_j|} \delta_k(t) \, c_k u_k \\ &= \mu_i^t \mu_j^t (v_1(t) + v_3(t)) + \mu_{l'}^t v_2(t) , \end{aligned} \tag{48}$$

providing[#1] $v_1(t) \neq 0$. $v_1(t)$ and $v_2(t)$ are bounded and $\|v_3(t)\|$ grows at most linearly. While any one of them can be zero, at least one of them is not equal to zero, if we assume that $l' \in \{1, \dots, n'\}$. Because the three vectors are linearly independent, we have $v_1(t) + v_2(t) + v_3(t) \neq 0$. Using this with the definition of the

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[#1] If $v_1(t) = 0$, if it happens that $v_3(t)$ is alternately zero and then nonzero, the limit of eq. (49) may not exist. However, from a physical point of view this is quite unlikely.

second order exponents we have now proved the following result:

$$\lambda_{ij}^{(2)} = \lambda^{(2)}(x, u_i, u_j) = \lim_{t \to \infty} \frac{1}{t} \log \|d_x^2 f^t(u_i, u_j)\|$$

$$= \max\{\lambda_i^{(1)}(x) + \lambda_j^{(1)}(x), \lambda_{l'}^{(1)}(x)\}, \quad (49)$$

with l' defined by (40). If (u_i, u_j) lies in the kernel of $d_x^2 f$ then $l' \notin \{1, \dots, n'\}$ and $\lambda_{i,j}^{(1)}$ will be equal to $-\infty$.

The extension of (49) to arbitrary vectors $w_i, w_j \in E_x^i \backslash E_x^{i+1}$ resp. $\in E_x^j \backslash E_x^{j+1}$ (in accordance with definition (29) is straightforward by using the bilinearity of $d_x^2 f^t$. The only possible difference might be the values of l', which is given by $d_x^2 f(w_i, w_j) \in E_x^{l'} \backslash E_x^{l'+1}$ and might be different to the l' given by $d_x^2 f(u_i, u_j)$ in degenerate cases.

For an arbitrary tangent vector $u \in E_x^l \backslash E_x^{l+1}$, where $d_x^2 f(u, u) \in E_x^{l'} \backslash E_x^{l'+1}$ this implies then

$$\lambda^{(2)}(x, u) = \max\{2\lambda_l^{(1)}(x), \lambda_{l'}^{(1)}(x)\}. \quad (50)$$

This last result can be understood intuitively in reference to eq. (33). u grows under $d_x f^m \otimes d_x f^m$ according to $2\lambda_l^{(1)}$. The second derivative $d_x^2 f$ generally maps u into another (possibly) more unstable direction l'. This is further mapped by the first term $d_{x_{m+1}} f^{t-1-m}$, which amplifies this according to $\lambda_{l'}^{(1)}(x)$. There is thus a competition between the two terms $2\lambda_l^{(1)}$ and $\lambda_{l'}^{(1)}(x)$ for dominance in the final growth rate.

For a *typical* fixed point, there is in general no reason to expect that $d_x^2 f$ preserves special directions, so that it will throw an arbitrary pair of tangent vectors onto $E_x^1 \backslash E_x^2$. Thus *typically* we expect that

$$\lambda_{ij}^{(2)}(x) = \max\{\lambda_i^{(1)}(x) + \lambda_j^{(1)}(x), \lambda_1^{(1)}(x)\}. \quad (51)$$

We should emphasize that these results assume that the second derivative (see (40)) is nonzero.

For a very high dimensional system, our conjecture implies that the leading first order exponent dominates the second order Lyapunov spec-

trum for large values of i and j. For example, if $\lambda_1^{(1)} > 2\lambda_2^{(1)}$, the diagonal part of the spectrum is of the form

$$\{\lambda_{ii}^{(2)}\} = (2\lambda_1^{(1)}, \lambda_1^{(1)}, \lambda_1^{(1)}, \dots). \quad (52)$$

Otherwise, if $2\lambda_2^{(1)} > \lambda_1^{(1)}$, but $\lambda_1^{(1)} > 2\lambda_3^{(1)}$, then the spectrum is of the form

$$\{\lambda_{ii}^{(2)}\} = (2\lambda_1^{(1)}, 2\lambda_2^{(1)}, \lambda_1^{(1)}, \lambda_1^{(1)}, \dots). \quad (53)$$

3.5. Periodic orbits

In the appendix we prove the same result we have already shown in one dimension, namely that

$$\lambda^{(q)}(f^p, x) = p\lambda^{(q)}(f, x). \quad (54)$$

3.6. Invariance properties

Since we have shown that the second order exponents can be expressed in terms of the first, we know that they share the same invariance properties.

4. Chaotic orbits

For chaotic orbits it is difficult to find simple examples where the higher order Lyapunov exponents can be computed by hand. The standard examples involving piecewise linear maps are uninteresting since the higher derivatives are identically zero almost everywhere, and so yield $\lambda_{ij}^{(2)} = -\infty$.

On chaotic attractors numerical experience and also some rigorous results indicate that there is a natural measure describing "typical" time averages so that $\{\lambda_i^{(1)}(x)\}$ takes on the same value for almost every x on the attractor. At special values of x, such as points on unstable periodic orbits, $\{\lambda_i^{(1)}(x)\}$ may take on different values, but these seem to be typically of zero measure.

There is a growing body of recent work [1], however, suggesting that in the limit as the period tends to infinity most unstable periodic orbits can be used to approximate the natural measure of the chaotic attractor they are contained in. For example, there is good numerical evidence that the first order Lyapunov exponents can be computed by simply finding as many long unstable periodic orbits as possible, computing their eigenvalues to find the Lyapunov spectrum of each orbit, and averaging over different orbits. We know of no rigorous proof that this procedure is correct, but in practice it seems to work very well.

The result suggested by these numerical studies is that, in the limit as the period goes to infinity, almost every unstable periodic orbit has statistical properties that are typical of those of the attractor it lives on. If a statement along these lines is true, then our results for the higher order Lyapunov exponents for periodic orbits should automatically extend to the chaotic orbits, with one modification: We must assume that there are no regions on the attractor of positive natural measure where the second derivative $d_x^2 f$ is identically zero. With this important restriction, we conjecture that equation (51) also holds

for chaotic orbits. Tom Taylor has recently proven an extension of Oseledec's theorem for higher order Lyapunov exponents (in one dimension), which puts this conjecture on firmer ground [7].

In the next section we offer numerical evidence suggesting that this conjecture is correct, at least for the leading second order exponent, $\lambda_{11}^{(2)}$.

5. Numerical results

The direct numerical computation of higher order Lyapunov exponents is more complicated than that of the usual first order exponents. The primary reason is that the higher order exponents cannot be written as a simple product of terms, and we cannot use the multiplicative ergodic theorem to average in the way that is customary for the first order exponent. Furthermore, there is no obvious procedure such as the usual Gram–Schmidt orthogonalization to compute the entire spectrum of higher order exponents. For the results reported here we confine ourselves to computations of the leading second order exponent, $\lambda_{11}^{(2)}$, which is easy to compute since, as we have shown, almost every pair of
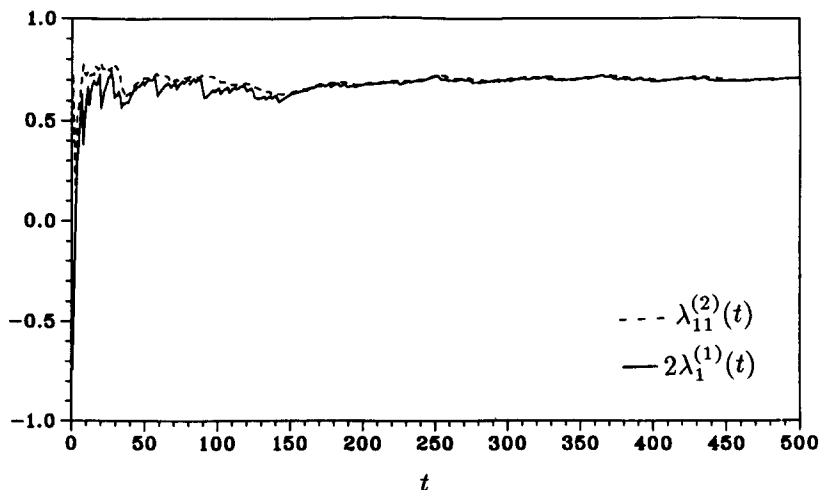

Fig. 1. Convergence of the largest second order Lyapunov exponent as a function of time, in comparison to twice the first order Lyapunov exponent, using the Ikeda map.

tangent vectors gives this exponent. Therefore we calculated $\lambda^{(2)}(x, u, t) = \log\|d_x^2 f'(u, u)\|/t$ for some arbitrary tangent vector u by implementing the recursion relation (30).

One practical difficulty is numerical overflow. When the dynamics are chaotic expressions such as eq. (33) quickly grow larger than the largest allowed value for the machine. This typically happens before we have iterated long enough to get a good numerical average over the attractor. For example, consider fig. 1, where we plot $\lambda^{(2)}(x, u, t)$ for an arbitrarily chosen x and u, where f is given by the Ikeda map

$$f(x, y) = (1 + \mu[x \cos(z) - y \sin(z)]\,,$$
$$\mu[x \sin(z) + y \cos(z)])\,, \tag{55}$$

with $z = z(x, y) = 0.4 - 0.6/(1 + x^2 + y^2)$ with $\mu = 0.7$ chosen.

For comparison, we have also shown as a dashed line $2\lambda^{(1)}(x, u, t)$. Although $\lambda^{(2)}(x, u, t)$ appears to converge to $2\lambda^{(1)}(x, u, t)$, this is inconclusive and does not provide very strong evidence for our conjecture.

We can considerably improve this situation by making use of a trick that allows us to get much
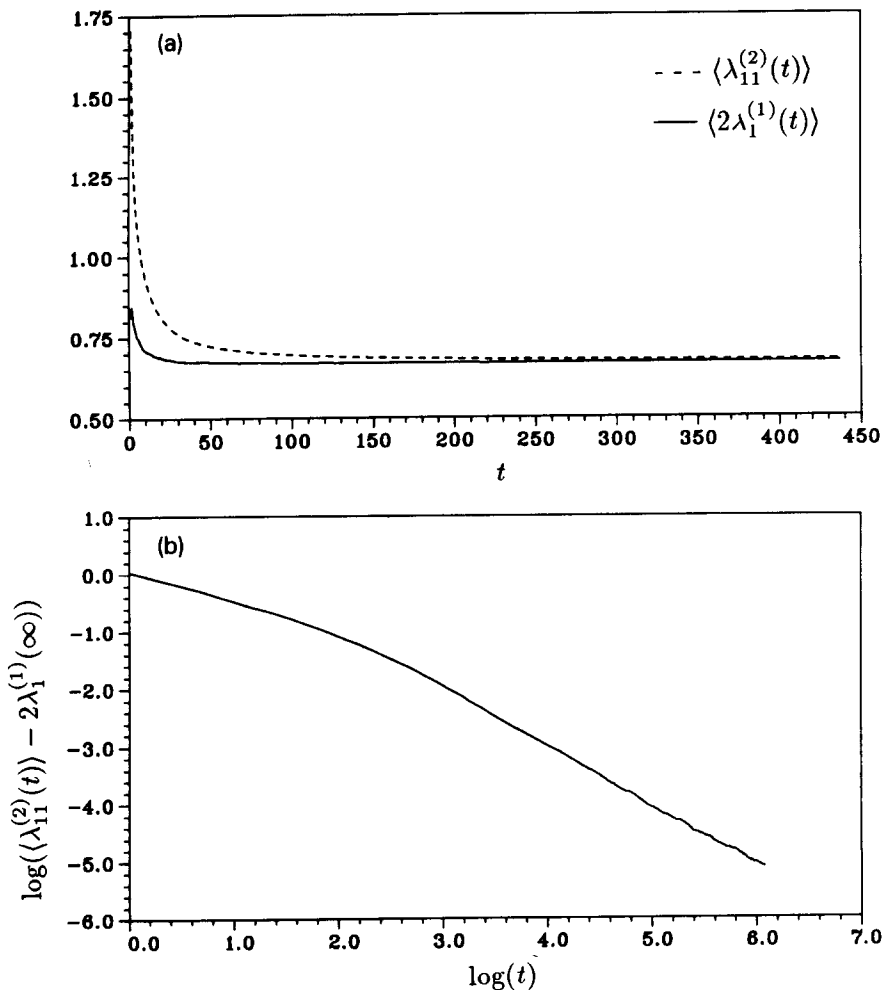

Fig. 2. (a) Same as fig. 1, but using the ensemble average technique of eq. (56). (b) plots log $(\langle \lambda_{11}^{(2)}(t) \rangle - 2\lambda_1^{(1)}(\infty))$ versus $\log(t)$, which makes the convergence clearer.

better numerical values. For an ergodic orbit with a natural measure almost every x yields the same Lyapunov spectrum $\lambda_{ij}^{(2)}$. Furthermore, almost every pair of tangent vectors (u, v) tends to $\lambda_{11}^{(2)}$. This implies

$$\lambda_{11}^{(2)} = \lim_{t \to \infty} \frac{1}{t} \langle \log \| d_x^2 f^t(u, v) \| \rangle_x \qquad (56)$$

where the average is taken over the natural measure.

We can take advantage of this and estimate $\lambda_{11}^{(2)}$ by computing

$$\langle \lambda_{11}^{(2)}(t) \rangle = \lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} \frac{1}{t} \log \| d_{x_j}^2 f^t(u, v) \| . \qquad (57)$$

We take N as large as we need to in order to get good statistical convergence, and make t the largest value that does not cause numerical overflow. This does not allow us to make t any larger, but it does mean that the estimates at each t are statistically stable. By extrapolating for several different values of t we can make sure that $\lambda_{11}^{(2)}(t)$ is converging to a well defined limit as $t \to \infty$.

In fig. 2 we repeat |the calculation of fig. 1 using the ensemble average technique of eq. (57). As seen in fig. 2a, the convergence is quite good. To provide a stronger test of this, in fig. 2b we plot $\log(\langle \lambda_{11}^{(2)}(t) \rangle - 2\lambda_1^{(1)}(\infty))$ versus $\log(t)$. This test is much stronger, since a small change from the true value destroys the monotonic convergence apparent in fig. 2b. We can thus say that $\lambda_{11}^{(2)} = 2\lambda_1^{(1)}$ to at least three digits of precision.

We have run similar experiments on several other standard dynamical systems, such as the logistic map, the Hénon map, and the "sine map", $x_{t+1} = a \cdot \sin(\pi x_t)$. In every case we have observed similar behavior, and our numerical experiments are in good agreement with $\lambda_{11}^{(2)} = 2\lambda_1^{(1)}$.

Acknowledgements

Appendix

Consider a periodic point $x = x_0$ of period p, i.e. $x_{n+p} = x_n$, $\forall n$. Any iterate m can be written $m = jp + r$ with $r \in [0, p-1]$. The sum in (4) can be written

$$\sum_{m=0}^{kp-1} = \sum_{j=0}^{k-1} \sum_{r=0}^{p-1} , \qquad (58)$$

which yields

$$x_{kp}'' = \sum_{j=0}^{k-1} \sum_{r=0}^{p-1} \sum_{n=jp+r+1}^{kp-1} f'(x_n) \cdot f''(x_{jp+r})$$
$$\cdot \left(\prod_{n=0}^{jp+r-1} f'(x_n) \right)^2 . \qquad (59)$$

The periodicity of x implies $x_{jp+r} = x_r$. Furthermore, since $d/dx\, f^k(x_n) = \Pi_{l=0}^{k-1} f'(x_{n+l})$, we can write

$$\prod_{n=jp+r+1}^{kp-1} f'(x_n) = \frac{d}{dx} f^{(k-j)p-r-1}(x_{jp+r+1})$$
$$= \prod_{l=r+1}^{p-1} f'(x_{(k-1)p+l})(x_p')^{k-j-1}$$
$$= (x_p')^{k-j-1} \cdot \prod_{l=r+1}^{p-1} f'(x_l) \qquad (60)$$

and

$$\left(\prod_{n=0}^{jp+r-1} f'(x_n)\right)^2 = (x_p')^{2j} \cdot \prod_{n=0}^{r-1} f'(x_n)^2 .$$ (61)

Using (60) and (61), x_{kp}'' yields

$$x_{kp}'' = \sum_{j=0}^{k-1} \sum_{r=0}^{p-1} (x_p')^{k-1-j}$$

$$\cdot \prod_{l=r+1}^{p-1} f'(x_l) \cdot f''(x_r) \cdot (x_p')^{2j} \prod_{n=0}^{r-1} f'(x_n)^2$$

$$= \sum_{j=0}^{k-1} (x_p')^{k-1+j}$$

$$\cdot \left(\sum_{r=0}^{p-1} f''(x_r) \prod_{l=r+1}^{p-1} f'(x_l) \prod_{n=0}^{r-1} f'(x_n)^2\right) .$$ (62)

Defining $c(x)$ as

$$c(x) = \sum_{r=0}^{p-1} f''(x_r) \prod_{l=r+1}^{p-1} f'(x_l) \prod_{n=0}^{r-1} f'(x_n)^2 ,$$ (63)

we finally obtain

$$x_{kp}'' = c(x) \cdot \sum_{j=0}^{k-1} (x_p')^{k-1+j} .$$

In higher dimensions the calculation is analogous to the 1D case. The crucial difference is that we have to preserve the ordering of the products because they generally do not commute.

Let x be a periodic point of period p of f, i.e. $x = f^p(x)$ and therefore $x_{jp+r} = x_r$. We split again the sum in (33) as in (58) and obtain

$$d_x^2 f^{kp} = \sum_{j=0}^{k-1} \sum_{r=0}^{p-1} d_{x_{jp+r+1}} f^{(k-j)p-1-r}$$

$$d_{x_{jp+r}}^2 f(d_x f^{jp+r} \otimes d_x f^{jp+r}) .$$ (64)

Now using

$$d_{x_{jp+r+1}} f^{(k-j)p-r} = (d_x f^p)^{k-j-1} d_{x_{r+1}} f^{p-r-1}$$ (65)

and

$$d_x f^{jp+r} = d_x f^r (d_x f^p)^j$$ (66)

we obtain

$$d_x^2 f^{kp} = \sum_{j=0}^{k-1} \sum_{r=0}^{p-1} (d_x f^p)^{k-j-1} d_{x_{r+1}} f^{p-r-1}$$

$$d_{x_r}^2 f(d_x f^r \otimes d_x f^r)$$

$$((d_x f^p)^j \otimes (d_x f^p)^j) .$$ (67)

Setting $A = d_x f^p$ and $B := \sum_{r=0}^{p-1} d_{x_{r+1}} f^{p-r-1} d_{x_r}^2 f(d_x f^r \otimes d_x f^r)$, then $d_x^2 f^{kp}$ turns out to be

$$d_x^2 f^{kp} = \sum_{j=0}^{k-1} A^{k-j-1} B(A^j \otimes A^j) .$$ (68)

This is essentially the same structure as formula (38) for a fixed point, except $d_x f$ is replaced by $d_x f^p$ and $d_x^2 f$ is replaced by the constant mapping B which is again more complicated than for a fixed point, but bounded. Now doing the same calculations as for a fixed point, we obtain exactly the same result:

Proposition. Consider eigendirections u_i, u_j of $A = d_x f^p$ with eigenvalue μ_i resp. μ_j, which are related to the first order Lyapunov exponents by $\log|\mu_i| = p\lambda_i^{(1)}(x)$ and let

$$B(u_i, u_j) = \sum_{k=l'}^{n'} c_k u_k;$$

then the second order Lyapunov exponent $\lambda^{(2)}(x, u_i, u_j)$ of a periodic point x and the directions u_i, u_j is given by

$$\lambda^{(2)}(x, u_i, u_j) = \max\{\lambda_i^{(1)}(x) + \lambda_j^{(1)}(x), \lambda_{l'}^{(1)}(x)\} .$$ (69)

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