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Studies of the limit order book around large price changes

Bence Tóth[1,2a], János Kertész[2], and J. Doyne Farmer[3]

[1] ISI Foundation - Viale S. Severo 65, 10133 Torino, Italy

[2] Institute of Physics, Budapest University of Technology and Economics - Budafoki út. 8. H-1111 Budapest, Hungary

[3] Santa Fe Institute, 1399 Hyde Park Road, Santa Fe, NM 87501

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Abstract. We study the dynamics of the limit order book of liquid stocks after experiencing large intra-day price changes. In the data we find large variations in several microscopical measures, e.g., the volatility the bid-ask spread, the bid-ask imbalance, the number of queuing limit orders, the activity (number and volume) of limit orders placed and canceled, etc. The relaxation of the quantities is generally very slow that can be described by a power law of exponent ≈ 0.4. We introduce a numerical model in order to understand the empirical results better. We find that with a zero intelligence deposition model of the order flow the empirical results can be reproduced qualitatively. This suggests that the slow relaxations might not be results of agents' strategic behaviour. Studying the difference between the exponents found empirically and numerically helps us to better identify the role of strategic behaviour in the phenomena.

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1 Introduction

Understanding the relaxation of a system to its typical state after an extreme event may give a lot of information on the dynamics of the system. Perhaps the first study of power law relaxations after extreme events was by Omori who was studying earthquake dynamics [1]. The Omori law describes the temporal decay of aftershock rates after a large earthquake: The number of aftershocks is described by a power law and no typical time scale of the relaxation can be found. Several further examples can be found for non exponential relaxations in complex systems: condensed state systems [2], spin glasses [3], microfracturing phenomena [4], internet traffic [5,6], etc. In case of financial markets Ref. [7] showed that volatility after market

[a] E-mail: bence@maxwell.phy.bme.hu

crashes can be also characterised by a power law decay, which is often referred to as the financial Omori law.

Our aim is to analyse large price changes that can happen often, maybe every month in case of liquid stocks. It is important to stress that we are not interested in market crashes or bubbles (when such changes happen throughout the market) but in large intra-day price changes specific to a particular stock. Understanding how the market relaxes after extreme events may be very important for volatility modeling and forecasting.

Refs. [8,9] analysed the post-event dynamics of large price changes appearing on short time scales for the New York Stock Exchange and Nasdaq. They found sharp peaks in the bid-ask spread, volatility and traded volume at the moment of the events with slow decay to normal value that in some cases could be characterised by a power law. Several studies dealt with the analysis of the structure of the limit order book preceding a large price change [10,11,12,13]. Their results show that the volume of market orders play a minor role in the creation of large price jumps. Instead it is the disappearance of liquidity in the limit order book that results in extreme price changes. Ref. [12] also studied the relaxation of the bid-ask spread after large spread variations. They found a slow relaxation to normal values, characterised by a power law with exponent around 0.4–0.5. Ref. [13] cross-correlate high-frequency time series of stock returns with different news feeds, showing evidence that news cannot explain the price jumps: In general jumps are followed by increased volatility, while news are followed by lower volatility levels.

In this paper we study the dynamics of the limit order book of the London Stock Exchange around large price changes. Analysing the limit order book allows us to look at the market at the level of single orders, and this way connect the microscopic dynamics to macroscopic measurables and possibly human behaviour. We focus on the post-event dynamics and the relaxation of the different measures.

We study the dynamics of the volatility and the bid-ask spread near large events. We find that both have peaks at the moment of the price change. Their relaxation is slow and can be characterized by a power law. Analysing the behaviour of market participants, we show results on the bid-ask imbalance, the number of queuing limit orders in the book, the aggregated number and aggregated volume of limit orders arriving, the aggregated number of cancelations and the relative rate of different order types. We find that the shape of the book and the relative imbalance changes very strongly, with a peak at the event and slow decay afterwards. The activity of both limit orders arriving and being canceled increases and, after a peak at the event, decays according to a power law. Surprisingly we find that the relative rates of limit orders, market orders and cancelations do not vary strongly in the vicinity of price jumps.

For the relaxation of most of the above measures we find power laws with similar exponents around $0.3-0.4$. The exponents are very similar, suggesting that there might be a common cause behind the slow relaxations of the different measures.

To further study the possible reasons for the relaxation of the volatility and the bid-ask spread, we construct a zero intelligence multi-agent model mimicking the actual trading mechanism and order flow. When introducing large price jumps in the model, we find slow relaxations in both of the values. This suggests that the slow relaxations can be reproduced without complicated behavioural assumptions. We show analytic results on relaxation of the spread in the zero intelligence model.

The paper is built up as follows: In Section 2 we review the continuous double auction mechanism and some properties of the limit order book. In Section 3 we present the data set and explain the method of determining large events. Section 4 shows our empirical results. In Section 5 we present our model and review the numerical and analytical results. We close the paper by summarising our results and present the conclusions in Section 6.

2 The limit order book

The market we are studying is governed by a continuous double auction. Continuous, since orders to buy or sell can be introduced any time to the market and are matched by an electronic system in the moment when a match becomes possible. In the market agents can place several different types of orders. These can be grouped into two main categories: *limit orders* and *market orders*.

Patient traders may submit limit orders to buy or sell a certain amount of shares of a given stock at a price not worse than a given limit price. Limit orders are not necessarily executed at the moment they are submitted. In this case they are stored in the queue of orders, the limit order book.

On the other hand impatient traders may put market orders, orders to buy or sell a certain amount of shares of a given stock at the best price available. Market orders usually are followed by an immediate transaction, matched to existing limit orders on the opposite side of the book according to the price and the arrival time.

The third important constituent of market dynamics are cancelations [14]: that is removing an existing limit order from the book. In general, limit orders increase liquidity, while market orders and cancelations decrease liquidity.

It is common to analyse *effective* limit and market orders, i.e. regard all orders that result in an immediate execution as market orders. This is right from the point of view of the effect of these orders. However, since we are interested in the behaviour of traders and in the way their order putting strategies may change, we do not use this notation, and define limit and market orders by their code in the limit order book, thus by the intention of the traders.

2.1 Notations

Buy limit orders are generally called *bids* and sell limit orders are generally called *asks*. At any time instant there exists a buy order with highest price (highest bid), b_t, and a sell order with lowest price (lowest ask), a_t, in the limit order book. The mean of the best bid and ask prices is the *mid-price*:

$$m_t = \frac{a_t + b_t}{2}. \quad (1)$$

We define volatility as the absolute change of the logarithm of the mid-price:

$$X_t = |\log m_t - \log m_{t-1}|, \quad (2)$$

where time is measured in minutes.

The difference between the logarithms of the lowest ask and the highest bid is called the *bid–ask spread*:

$$S_t = \log a_t - \log b_t. \quad (3)$$

This gives a measure of transaction costs in the market (and on the other hand the profit of market making strategies).

Further important factors in the limit order book are the *gaps*, i.e. the number of adjacent unoccupied price levels between existing limit orders. Most often one talks about the first gap, defined as the difference between the best log-price and the next best log-price in either side of the book:

$$g_t^{(1)} = |\log p_t^{best} - \log p_t^{next}|. \quad (4)$$

The gap is one tick if adjacent price levels are filled. The second ($g^{(2)}$), third ($g^{(3)}$), etc. gaps can be defined in a straightforward way.

3 Data and methodology

3.1 The data set

We studied the data of 12 liquid stocks of the London Stock Exchange for the period 05.2000 to 12.2002. The stocks studied were: Astrazeneca (AZN), Baa (BAA), Boots Group (BOOT), British Sky Broadcasting Group (BSY), Hilton Group (HG.), Kelly Group Ltd. (KEL), Lloyds Tsb Group (LLOY), Prudential (PRU), Pearson (PSON), Rio Tinto (RIO), Shell Transport & Trading Co. (SHEL), Vodafone Group (VOD).

The London Stock Exchange consists of two parts: the downstairs market (SETS) and the upstairs market (SEAQ). The downstairs market is governed by a fully automatic order matching system, while the upstairs market is arranged informally through direct connections between agents. We confine our study to the electronic downstairs market. On the LSE there are no official market makers, instead every trader can act as a market maker anytime by posting bid and ask orders simultaneously. For more details on the rules of the LSE see [15].

3.2 What are large events?

To study the limit order book dynamics around large events, we first have to define a filter that determines not only large price changes but also the moment of the event in a consistent manner. When determining the events, one has to face the following problem: there are volatile stocks for which even a price change of 3-4 % can be an everyday event, while for some less volatile stocks a much smaller price change can be the sign of a major event. In filtering for large events, we are going to follow the method proposed by [9].

In order to determine the events, we use two filters on our data.

1. *Absolute filter* The first filter searches for intra-day price changes larger than 2 % of the current price in time windows not longer than 120 minutes.
2. *Relative filter* We measure the average intra-day volatility pattern for the stock in the period prior to the event. The filter searches for intra-day price changes in time windows not longer than 120 minutes, exceeding 6 times the normal volatility during that period of the day.

Normal volatility is defined as the average volatility for the same period of the day computed over the 60 preceding days. A price change is considered an event if it passes both of the above filters. When looking for large events we use transaction prices (both for returns and volatility) and the price change is understood to be change in the log-price. Furthermore we omit the first 5 minutes of the trading day to exclude opening effects from our measurements. We also omit the last 60 minutes of the trading day in order to be able to study the intra-day after event dynamics. The method is quite robust when altering the threshold values.

To be able to localize the exact moment of the events we look at the earliest and shortest of the time windows in which both of the thresholds have been exceeded. This means that, when looking for 120 minute events, if the price change already passes the filters in, say, 42 minutes, then we assume the price change has taken place in 42 minutes and the end of the time window is set to the earliest moment by which the thresholds have been exceeded. The end of the time window in which the event took place is considered the end of the price change and from this point we start studying the post event dynamics. This way the minute 0 of the event is exactly the end of the time window.

With the filters defined above we were able to determine 289 events for the 12 stocks in the roughly 2.5-year period. We found a total of 169 downward events and 120 upward events.

4 Empirical results

We studied the dynamics of several measures of the order book. In Figure 1 we show the changes in the volatility, the bid-ask spread and the number of limit orders placed and canceled. All plots show the dynamics for the 60-minute pre-event and 120-minute post-event periods. For all events we defined time 0 as the end of the shortest and earliest time windows in which the filters defined in Section 3.2 have been passed. Then, in case of each event we compared the dynamics in the period from -60 minutes to 120 minutes to the average volatility dynamics for the same period of the day computed over the 60 days preceding the event. In the next step we aggregated the dynamics over the events. This way all the plots show dynamics compared to the average dynamics, daily periodicities excluded.

Figure 1(a) shows the dynamics of the volatility and Figure 1(b) shows the dynamics of the spread. In this case we included both downward and upward price jumps in our sample. Figure 1(c) and 1(d) show how the number of limit orders placed has changed on the bid side and the ask side of the book in case of downward price jumps. Figure 1(e) and 1(f) show how the number of limit orders canceled has changed on the bid side and the ask side of the book in case of downward price jumps. The number of orders placed and canceled may be regarded as measures of the activity of market participants. (In case of the activities, the plots for upward price jumps are very similar to those with downward jumps, so we only show the negative events.)

We find that the dynamics are very similar. All measures have a strong peak at the moment of the event determined by our filter. We are going to focus on the relaxation of the variables after the peak: This is the part that is well defined by our filtering method. Concerning the *rise* prior to the events we have to be careful, however. Due to our method of determining events (the fact that we define the end of the time window as zero time of the event) the pre-event dynamics are conditioned on the event and can not be regarded as independent. Practically, what we can say is that the variables change near the event with a peak at the moment of the event, but the actual rise can not be characterised through these results. The relaxations after the event seem to be slow in all cases. In Figure 2 we show the relaxation of the same measures after the event on a log-log scale. In order to quantify the relaxations we apply power law fits to the curves and compare the power law exponents. (Note that to study the relaxations to the normal value, we plot the excess variables, thus the difference between the actual value and the value in normal periods.) In Table 1 we summarise the exponents of the relaxations.

As we can see the *volatility* increases very high, to roughly 12 times its normal value and its decay can be well fit by a power law with an exponent of approximately 0.38. The result on the power law relaxation with exponent 0.38 of the excess volatility after events can be compared to results on the relaxation after crashes. Ref. [7] shows that volatility after stock market crashes decays approximately with an exponent of 0.3, showing that the post-crash dynamics are similar to those of earthquakes, commonly known as the Omori law. Another study showed that the relaxation of excess volatility after fluctuations is characterized by a power law with exponent between 0.3-0.4, measured to be robust for different time periods and across markets [16]. This may point us to the fact that relaxation of the volatility after extreme events and after fluctuations are similar.

The *bid-ask spread* increases, but to a lower value than the volatility, with a peak of roughly 3 times the average. The first part of the relaxation curve seems to show scaling behaviour and may be fit by a power law. Interestingly, we find that the exponent of the decay is around 0.38, very close to the exponent of the volatility decay. The slow, power law like decay of excess spread is in agreement with the findings of [12]. However the scaling for the bid-ask

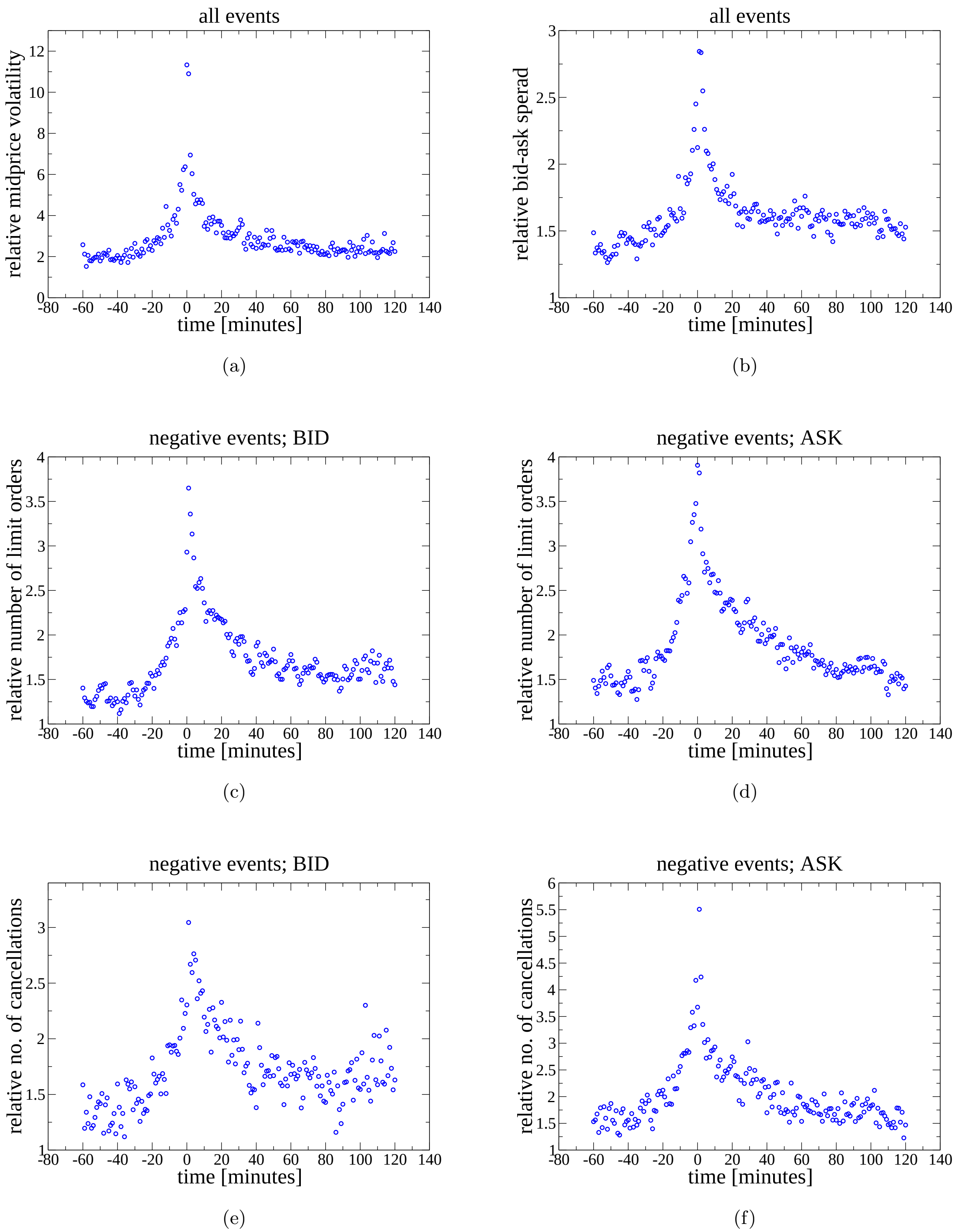
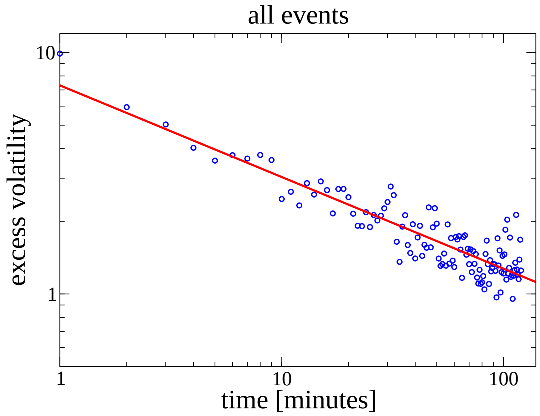
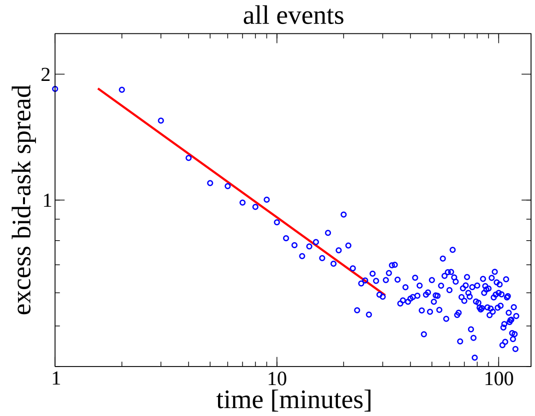


Fig. 1. (Color online) Dynamics around events compared to the usual dynamics. Usual dynamics are computed from the 60-day pre-event average for the same period of the day. (a) Mid-price volatility averaged over all events; (b) Bid-ask spread averaged over all events; (c) The number of limit orders in case of negative events on the bid side of the book and (d) on the ask side of the book; (e) The number of limit order cancelations in case of negative events on the bid side of the book and (f) on the ask side of the book.



(a)



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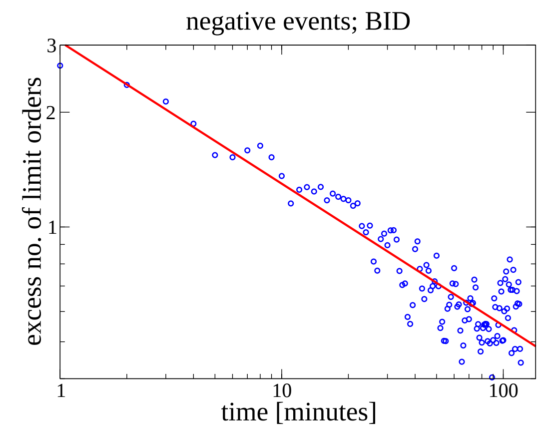
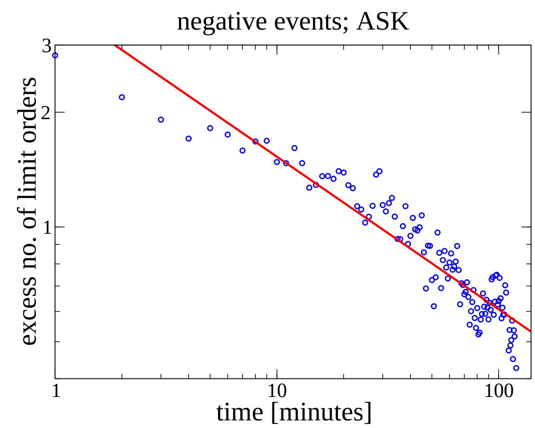


(c)



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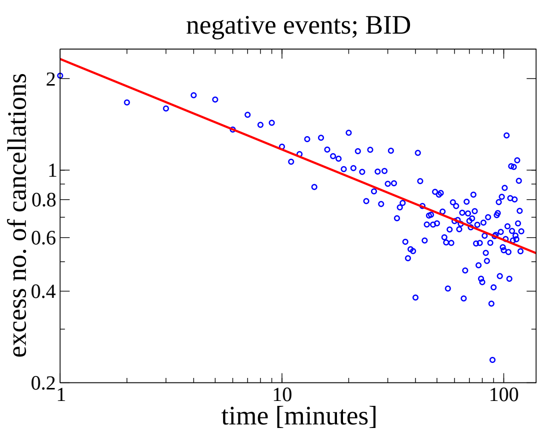
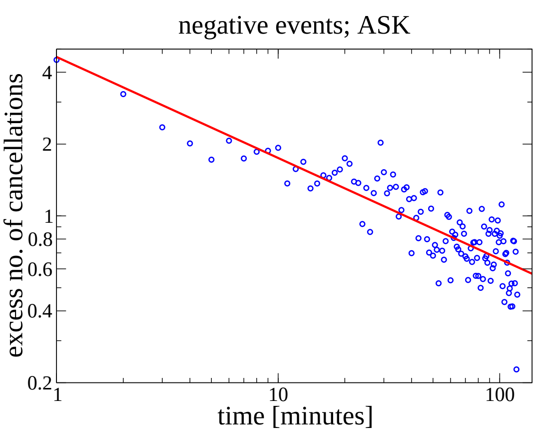


(e)



(f)

Fig. 2. (Color online) Relaxation of the excess variables on log-log scale with power law fits: (a) Mid-price volatility averaged over all events, exponent: 0.38 ± 0.01; (b) Bid-ask spread averaged over all events. The first part of the relaxation curve seems to show scaling behaviour and may be fit by a power law with an exponent: 0.38 ± 0.03. However the scaling for the bid-ask spread is less consistent than that of the volatility. Also the long time decay is apparently even slower than a power law.; (c) The number of limit orders in case of negative events on the bid side of the book, exponent: 0.37 ± 0.01, and (d) on the ask side

spread is less consistent than that of the volatility. Also the long time decay is apparently even slower than a power law. Interestingly, [9] found strong variation of the spread after large events at NYSE, but not on Nasdaq. They argued that this difference is caused by the diverse trading rules on the two markets, pointing out that the existence of a single market maker at NYSE leads to the opening of the spread. The trading rules of LSE, being a fully automatic market makes it similar to Nasdaq, so, according to that argument, one would expect low variation of the spread on the LSE. This is not consistent with our findings. An explanation of this contradicting behaviour on the two markets is missing at present.

The *number of limit orders placed* increases both on the bid and the ask side of the book. We find a peak of roughly four times the usual value with a slow decay of the activity afterwards. The relaxation of the excess limit order putting activity after events can be described by a power law decay with an exponent of roughly 0.37–0.4. Concerning the dynamics of the aggregated *volume of limit orders placed* we get very similar results (not shown here). However, in case of the volumes we see a slightly stronger increase around the event with roughly 6 times the usual value. This suggests that the average volume of limit orders increases around large price changes. The relaxation of the excess limit order volume can be characterised by a power law of exponent roughly 0.44 in case of negative events and exponent 0.48 for positive events.

Similarly, the *number of limit orders canceled* increases strongly around events, with a slow decay after the peak.

Table 1. The exponents of the relaxation curves for the different variables.

variable	exponent
volatility	0.38 ± 0.01
bid-ask spread	0.38 ± 0.03
limit orders placed - bid	0.37 ± 0.01
limit orders placed - ask	0.40 ± 0.01
cancelations - bid	0.30 ± 0.02
cancelations - ask	0.42 ± 0.02

It also seems that the increase is stronger and much more clear on the side of the book which is opposite to the direction of the price change, i.e. ask side in case of negative events and bid side in case of positive events. For the relaxations we find that in general, the decays on the opposite side of the book compared to the direction of the price change can be well fit by a power law of exponent roughly 0.42. The decays on the same side as the price change direction are more noisy but may be fit by a power law, showing exponents of roughly 0.3–0.35.

It is generally understood that the relative amount of supply and demand govern the movement of prices. To quantify the pressure of orders from either side of the book, we study the *bid-ask imbalance* on the market. We denote the total volume to buy on the market by V_t^{buy} and the total volume to sell by V_t^{sell}. Then we define the buy imbalance and the sell imbalance as:

$$I_t^{buy} = \frac{V_t^{buy}}{V_t^{buy} + V_t^{sell}} \tag{5}$$

and

$$I_t^{sell} = \frac{V_t^{sell}}{V_t^{buy} + V_t^{sell}}. \tag{6}$$

Trivially, $I_t^{buy} + I_t^{sell} \equiv 1$.

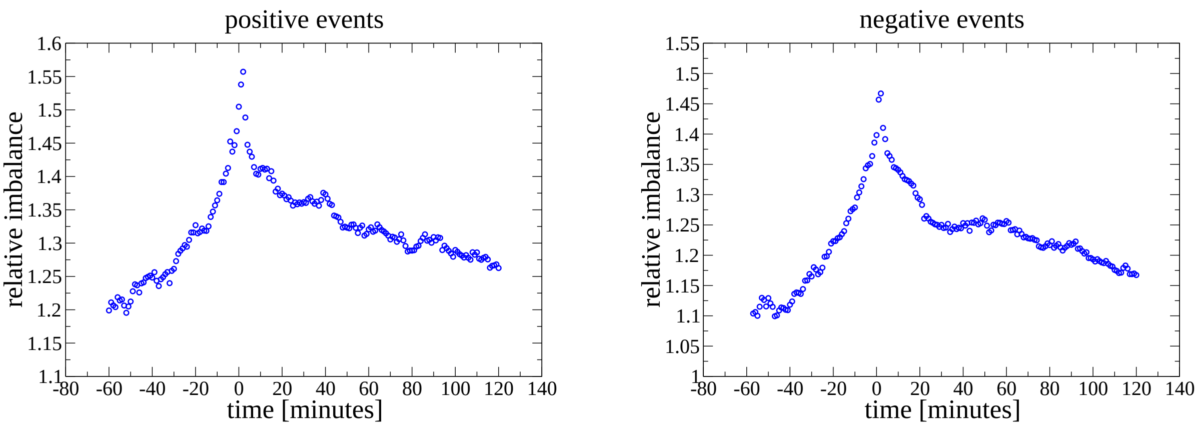
In Figure 3 we show the dynamics of the imbalances for upward and downward price changes, compared to the 60 day pre-event interval for the same period of the day, as before. The two curves are very similar showing a vanishing amount of orders to sell in case of upward price moves and vanishing amount of orders to buy in case of downward price moves.

To understand the absolute values on the y-axis is not straightforward, to make it clear we show the following: If we assume that in regular market periods, on average half of the volume of orders is to buy and consequently half is to sell, then the values on the y-axis show that in case of the moment of a positive event, only $\approx 100\% - 50\% * 1.57 = 21.5\%$ of the total volume of orders appears on the sell side of the book and in case of the moment of a negative event, only $\approx 100\% - 50\% * 1.48 = 26\%$ of the total volume of orders appears on the buy side of the book. These numbers show that there is a huge imbalance of volume around events. The relaxation of the imbalance after the events is very slow.

A measure very similar to the bid-ask imbalance, is the *number of queuing orders* on either side of the limit order book. Figure 4(a) shows the dynamics of the number of queuing orders in the limit order book compared to the 60 day pre-event interval for the same period of the day, for negative events. The plot shows results for both the bid and the ask sides of the book. The results for positive events are very similar (of course symmetrically to the case of negative events), so we only show the figures for the negative events.

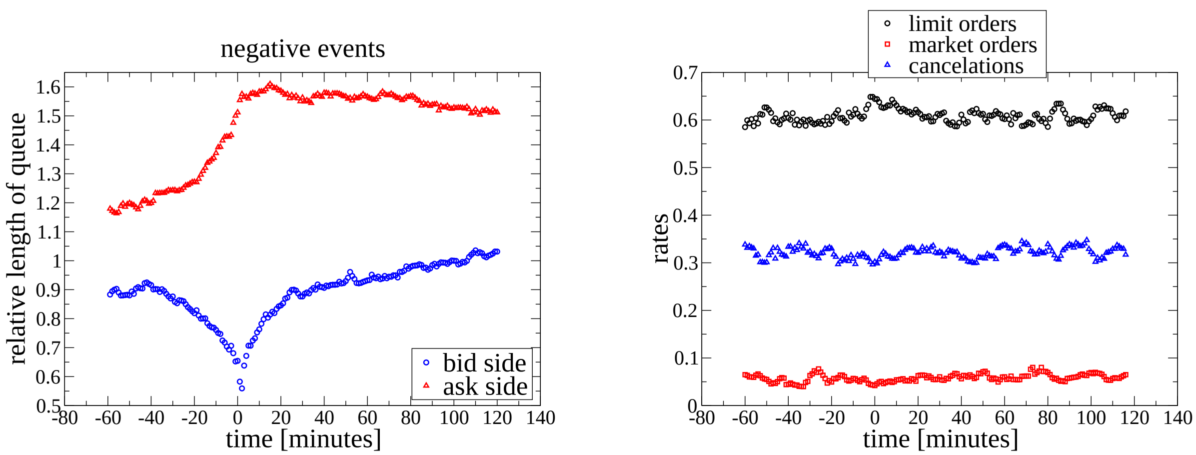
We can see that for negative price jumps the number of queuing limit orders on the bid side decreases to about half of the usual value and only relaxes back slowly. At the same time on the ask side the number of queuing orders increases to roughly 1.6 times the usual value and even after the event stays very high for a long time. Further studies are needed to see if this very slow relaxation is a sign of the limit order book being partly frozen in post-event periods.

It is interesting to study the rates of different market activities, i.e. the relative number of limit orders, market orders and cancelations, compared to the total number of orders. Figure 4(b) presents the dynamics of the three rates around negative events for the buy side of the book. We see that there are no strong changes in the rate around large price changes. The results for the ask side of the book are the same. When studying positive events we get very similar dynamics: No strong variation in the relative rates. This result means that it is rather the entire market activity changing (increasing) in the surroundings



(a) (b)

Fig. 3. (Color online) The dynamics of the imbalance in the volume of supply and demand: (a) shows the buy imbalance (I^{buy}) in case of upward price jumps; (b) shows the sell imbalance (I^{sell}) in case of downward price jumps.



(a) (b)

Fig. 4. (Color online) (a) Dynamics of the number of queuing orders in the book in case of negative events. The number of queuing limit orders on the bid side decreases to about half of the usual value and only relaxes back slowly. The number of queuing limit orders on the ask side increases to roughly 1.6 times the usual value and even after the event stays very high for a long time.; (b) The dynamics of the rates of limit orders (black circles), market orders (red squares) and cancelations (blue triangles) around negative events for the buy side of the book. No very strong variations can be seen in the different rates around the large price changes.

of large events and not the strategy of traders how they place orders.[1]

[1] As we stated before, we regard orders as limit or market orders by the intention of the trader, not by their effect (i.e. not effective limit and market orders). Because of this we get a higher rate of limit orders and lower rate of market orders, than presented in [10]. However the rate of cancelations fits their results very well, so we believe that after accounting for

Summarising the empirical results: We have studied the dynamics of several measures of the order book before and after large price changes. For the change in the volatility, the bid-ask spread, the limit order placing and cancelation activity, the bid-ask imbalance, the number of queuing orders in the book we found strong variation at the moment of the event and slow relaxation in the post-event period. Specially, in case of the volatility, the spread and the activities we found a relaxation very similar to a power law, with exponents close to 0.4 suggesting a possible common cause behind the slow relaxations of the different measures.

Analysing the rates of limit orders, market orders and cancelations around large events, we did not find strong variation, showing that strategy of traders in choosing their type of order does not vary much.

5 An agent-based model

As stated above, we found very similar relaxation in different measures of the limit order book. To better understand the dynamics leading to the slow relaxations, in this section we introduce a multi-agent model of the order placing and removing process. When constructing a modeling framework, we have to decide which path to follow:

1. Building a multi-agent model with complicated strategies, involving behavioural assumptions.
2. Building a *zero intelligence* multi-agent model, with some very basic assumptions on the order flow.

the effective orders, there are no contradictions between the two results.

In the literature there are several examples for both types of models. Models of type 1 permit one to study behavioural results of the model, but when building the trading strategies we have to be careful, not to assume unrealistic properties of traders and/or avoid the common error, to simply find as output exactly the input assumptions. Models of type 2 are easier to construct, but apart from the problem of possible over-simplification, they also confine us to the analysis of non-behavioural measures through the model.

We chose to follow the path of [17,18,19,20,21,22,23,24] to construct a zero intelligence multi agent model of the continuous double auction through the limit order book. The model is aimed to be as simple as possible but capturing the most important properties of the continuous double auction. If we are able to reproduce some results with a zero intelligence model it may suggest that the particular phenomenon is not due to traders' strategic behaviour but rather to the market mechanism or institutions.

5.1 Details of the model

We assume limit order placing and cancelations similarly to a deposition–evaporation process and furthermore introduce market orders. Our model is similar to Maslov's model [19]. The main differences are that we allow for cancelation of existing limit orders (and through this the tuning of the probabilities of different actions) and that agents put their limit orders relative to the mid-price, this way we allow for a non trivial dynamics of the bid-ask spread. All orders arrive or evaporate with the same unit volume. The trading mechanism is the following:

– Limit orders arrive with rate P_{LO} per unit time with equal probability to buy or sell. Limit orders get deposited in the interval $[m_t - D, m_t]$ in case of buy orders and in the interval $[m_t, m_t + D]$ in case of sell orders with uniform distribution, where m_t is the mid-quote price (see Equation 1) and D is a parameter of the model.
– Market orders arrive with rate P_{MO} per unit time with equal probability to 'buy' or 'sell'. A market order to buy (sell) will get executed immediately by being matched to the best limit order to sell (buy).
– Existing limit orders are being canceled with rate P_C per unit time from the 'buy' or 'sell' side with equal probability. In case of a cancelation on one side of the book, all limit orders on that side have the same probability: $p = V_{total}^{-1}$ to be canceled where V_{total} is the total volume of limit orders on that side of the book. Thus on average one limit order evaporates from the book in case of cancelations.

The three rates add up to one: $P_{LO} + P_{MO} + P_C = 1$ (in other words we study the model in event time). In the beginning of the simulation there is a long warming up period, when orders are randomly placed in the book, in order not to have spurious results due to fluctuations and an empty book. Figure 5 shows the scheme of the order flow mechanism. Since in our model there are no crossing limit orders, in order to have a stationary number of orders we set $P_{LO} = 0.5$.

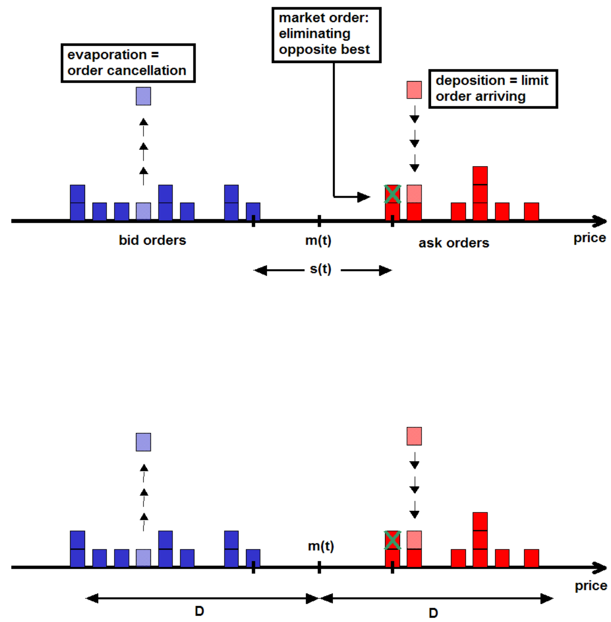


Fig. 5. (Color online) Scheme of the dynamics of the zero intelligence model.

We generate large price changes manually, by clearing out all limit orders on one side of the book[2] in the interval $[b_t - J, b_t]$ for price drops and $[a_t, a_t + J]$ for price jumps, where J is a parameter of the model (and b_t and a_t are the best bid and best ask respectively), this way the jump in the mid-price is $J/2$.

As we have mentioned, with such a simple model we have to confine ourselves to the study of the bid-ask spread and the volatility, i.e., variables whose dynamics may be

[2] This method is clearly not realistic. However, this is the only way we could think to generate large price changes in a zero intelligence model, i.e., when price jumps are not governed by sudden changes in the agents perception, strategy and herding behaviour. Since we are interested in the relaxation after the jump, we believe that this way of generating the large price changes is satisfactory.

studied in a model without adaptation rules and agents' intelligence (unlike other variables, like the activity, that are more closely related to changes in agents' perception and thus cannot be analysed in a simple zero intelligence model).

5.2 Numerical results

For usual market dynamics the rate of different orders are roughly $P_{LO} = 0.5$, $P_{MO} = 0.16$ and $P_C = 0.34$ [10][3]. In this section we present the results of our numerical simulations using the above empirical probabilities. The parameters of the simulation are the following:

- $D = 1000$
- $J = 1000$
- the frequency of large price jumps was $f = (5 * 10^4)^{-1}$
- the length of the simulation was $5 * 10^6$.

Figure 6 shows the average decay of the volatility and the bid-ask spread after large price changes in the model. The plots are created in the manner as the figures in Section 4: Time zero is the moment of the event and the y-axis shows the relative dynamics compared to stationary market periods, averaged over 100 events. The decays are qualitatively similar to what we have seen for the empirical data. The short time relaxations (up to roughly 100 simulation steps) can be described by power laws, with exponents of roughly 0.5 (0.50 ± 0.04 for the volatility and 0.48 ± 0.01 for the bid-ask spread). However, there is a difference between the empirical and numerical exponents. This suggests that our simple model is not able to entirely reproduce the relaxations. It seems that part of the slow relaxation can be generated in the model, however the difference between the exponents 0.4 and 0.5 is important and we believe that the discrepancy is due to the behaviour of the agents, that can not be captured in the zero intelligence model. It is interesting that the exponent of the decay in the volatility and in the bid-ask spread seem to be very close to each other, similarly as in the case of the empirical data. Also similarly to the empirical data, the peak in the volatility is smaller than that in the bid-ask spread.

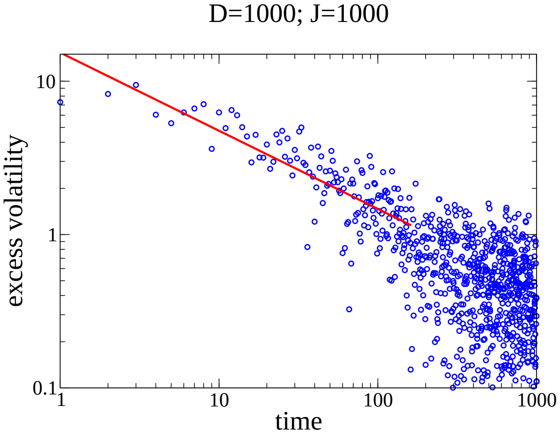
[3] Note that these rates are defined for *effective* limit orders and *effective* market orders, i.e. all orders that lead to an immediate execution are regarded as market orders. Since we do not have crossing limit orders in our simulation, it is right to use these values.

5.3 Analytical treatment

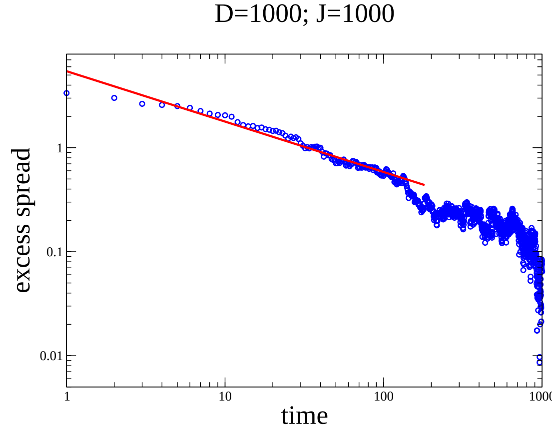
As we have seen, the bid-ask spread shows a slow relaxation both for empirical data and simulations. To understand if it is really a critical relaxation and what determines the exponent, we tried to treat the model analytically.

5.3.1 A limit case

The model can be treated analytically in the case of $P_{MO} = 0$, i.e. when only limit order placing and cancelation determine the flow ($P_{LO} = P_C = 0.5$), using some simple assumptions.



(a)



(b)

Fig. 6. (Color online) Relaxations after price jumps in the numerical model, in case of $P_{LO} = 0.5$, $P_{MO} = 0.16$ and $P_C = 0.34$. We also show the power law fits of the relaxations. (a) relaxation of the volatility, 0.50 ± 0.04; (b) relaxation of the bid-ask spread, 0.48 ± 0.01; Both exponents are close to 0.5. Similarly to empirical results, the variation in the volatility is much stronger than in the spread.

Mathematically the spread in this limit case can be understood as the time evolution of the minimum of numbers distributed uniformly on a finite interval (except for very short times). This tells us that the spread will decay according to a power law with unit exponent, clearly showing that the limit case is unrealistic. Nevertheless we discuss it briefly as we are going to use these results in the following.

When describing the time evolution of the spread we use a simplifying assumption: Since the probability of the best order to be canceled on either side is very low, we can assume that cancelations do not alter the value of the spread. With the above assumption, the expectation of the change in spread from one time step to another can be written in the following way:

$$\mathbb{E}(\Delta S_t) = P_C \cdot 0 + P_{LO}\Big(\frac{D - \frac{S_t}{2}}{D} \cdot 0 - \frac{1}{D}\sum_{k=1}^{\frac{S_t}{2}} k\Big), \qquad (7)$$

where the first term on the right hand side assumes that cancelations do not change the spread, and the second term describes the effect of a new limit order arriving: If it falls outside the spread it does not change the spread, if it falls inside the spread with a distance k from the same best price, it decreases the spread exactly by k. Summing up the right side of Eq. 7 we get

$$\mathbb{E}(\Delta S_t) = -P_{LO}\Big(\frac{S_t^2}{8D} + \frac{S_t}{4D}\Big). \qquad (8)$$

Using Equation 8, in case of $P_{LO} = 0.5$ the expected value of the spread can be written in the following recursive formula:

$$\mathbb{E}(S_{t+1}) = \mathbb{E}(S_t) + \mathbb{E}(\Delta S_t) = \\ = \mathbb{E}(S_t)\left(1 - \frac{1}{8D}\right) - \frac{\mathbb{E}(S_t)^2}{16D}. \quad (9)$$

The above recursive formula is a mean field theory that can be used to describe the relaxation when knowing the initial spread after the event. The formula fits the numerical results very well reproducing the asymptotic power law with exponent very close to 1.

5.3.2 General case

We are most interested in the general case when the probability of market orders is finite. For $P_{MO} \neq 0$, the change of the spread is the following:

$$\mathbb{E}(\Delta S_t) = P_C \cdot 0 + P_{LO}\left(\frac{D - \frac{S_t}{2}}{D} \cdot 0 - \frac{1}{D}\sum_{k=1}^{\frac{S_t}{2}} k\right) + \\ + P_{MO} \cdot g_t^{(1)} = -P_{LO}\left[\frac{S_t^2}{8D} + \frac{S_t}{4D}\right] + P_{MO} \cdot g_t^{(1)}, \quad (10)$$

where the first term assumes again that cancelations do not change the spread, the second term gives the expected change in case of a limit order arriving (similarly to the limit case, Eq. 7) and the last term stands for a market order arriving, increasing the spread exactly by the size of the first gap, $g_t^{(1)}$. As we can see, when having market orders, not surprisingly we have to account for the gaps as well. The expectation for the gap can be given by the following equation:

$$\mathbb{E}(g_t^{(1)}) = P_C \cdot g_{t-1}^{(1)} + \\ + P_{LO}\left[\frac{D - \frac{S_t}{2} - g_{t-1}^{(1)}}{D} g_{t-1}^{(1)} + \frac{1}{D}\sum_{k=1}^{g_{t-1}^{(1)}} k + \right. \\ \left. + \frac{1}{D}\sum_{k=1}^{\frac{S_{t-1}}{2}} k\right] + P_{MO} \cdot g_{t-1}^{(2)}, \quad (11)$$

where $g^{(2)}$ stands for the second gap in the book, i.e. the price difference between the second best and third best order on the same side of the book. In Equations 10 and 11 we neglected the probability of cancelations changing the spread or the first gap. As we can see, in the expected value of the first gap, a term containing the second gap occurs. This is the general case for all gaps that is, when writing up $g^{(n)}$, it will contain a term depending on $g^{(n+1)}$. We have to find a closure for this infinite hierarchy of equations, so we do the following. First, we estimate the relation between the first and the second gaps for the equilibrium state of the system (the value of the second gap, as a function of the first gap). Second, we assume this relation to be constant also for the relaxation period. By equilibrium state, here we mean the long time behaviour of the system without large price jumps. We denote the equilibrium values of the spread, the first gap and the second gap by σ, $\gamma^{(1)}$ and $\gamma^{(2)}$ respectively. In case of stationarity, $\mathbb{E}(\Delta S_t) = 0$, thus Equation 10 becomes:

$$P_{LO}\left[\frac{\sigma^2}{8D} + \frac{\sigma}{4D}\right] = P_{MO} \cdot \gamma^{(1)}. \quad (12)$$

Equation 11 becomes

$$\gamma^{(1)} = P_C \cdot \gamma^{(1)} + \\ + P_{LO}\left[\frac{D - \frac{\sigma}{2} - \gamma^{(1)}}{D}\gamma^{(1)} + \frac{1}{D}\sum_{k=1}^{\gamma^{(1)}} k + \right. \\ \left. + \frac{1}{D}\sum_{k=1}^{\frac{\sigma}{2}} k\right] + P_{MO}\cdot\gamma^{(2)}. \quad (13)$$

Combining Equations 12 and 13, we get the following formula:

$$\frac{\gamma^{(2)}}{\gamma^{(1)}} = 1 + \frac{1}{2D}\frac{P_{LO}}{P_{MO}}\left[\sigma + \gamma^{(1)} - 1\right] - 2P_{LO}. \quad (14)$$

Knowing that $P_{LO} = 0.5$ in the model, we get

$$\frac{\gamma^{(2)}}{\gamma^{(1)}} = \frac{1}{2D}\frac{P_{LO}}{P_{MO}}\left[\sigma + \gamma^{(1)} - 1\right]. \quad (15)$$

The above relation between the first and second gap can be assumed to be true for the relaxation process, since the second gap has only a minor role in case of decreasing spread. Introducing Equation 15 into Equation 11, we get a recursive formula for the size of the first gap, and through that for the size of the spread. To be able to use the relations, we need to know the expected value of the stationary spread, σ. This we know from the decay of the spread: The value, of the first point (the relative opening of the spread) is approximately $(\sigma + J)/\sigma$.

Figure 7 shows the comparison of the numerical and the analytical results for the relaxation of the spread for $P_{MO} = 0.16$.

The analytical formula seems to describe the relaxation of the spread for short times but not long times. We can see that the agreement between the analytical and numerical

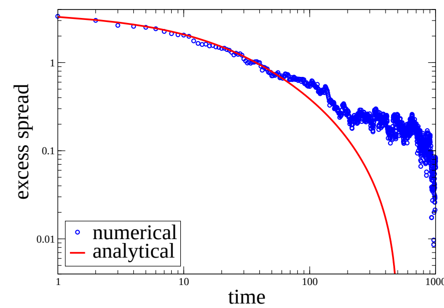


Fig. 7. (Color online) The relaxation of the spread after large price changes in case of $P_{MO} = 0.16$. The black circles show the numerical values, the red line shows the analytical values.

curves only hold for roughly the first 100 simulation steps in the relaxation. This is due to the fact that on long times the "mean field" assumption ignoring the actual dynamics of the second gap gives rise to larger errors.

6 Conclusions

We have studied the dynamics of several measures of the order book after large price changes. For the change in the volatility, the bid-ask spread, the limit order placing and cancelation activity, the bid-ask imbalance, the number of queuing orders in the book we found strong variation at the moment of the event and slow relaxation in the post-event period. Specially, in case of the volatility, the spread and the activities we found a relaxation very similar to a power law, with exponents close to 0.4.

Analysing the rates of limit orders, market orders and cancelations around large events, we did not find strong variation, showing that strategy of traders in choosing their type of order does not vary much.

To deeper understand the similar slow relaxations found empirically, we constructed a zero intelligence multi agent model for the order flow. The model is essentially a deposition-evaporation model with market orders added. The large price changes were generated manually. We found that the simple model was able to reproduce the relaxation of volatility and bid-ask spread qualitatively. The relaxations in the model were slow, similar to a power law with exponents very close to each other, similarly to the case of empirical data. The value of the exponents found was roughly 0.5 both for the volatility and the bid-ask spread, slightly higher than empirically. The ratio of the peak in the volatility and in the spread was also similar to the one found empirically. Consequently, though we find that the relaxations are slower in real markets than in the simulations, the values suggest that the overall character of the slow relaxations can be explained in the framework of the zero intelligence model without assuming strategic behaviour of agents. However the difference between the exponents 0.4 and 0.5 is important and we believe that the discrepancy is due to the behaviour of the agents, that can not be captured in the zero intelligence model.

We gave an analytic solution for the relaxation of the bid-ask spread in the model for the limit case of $P_{MO} = 0$ and for the short term relaxation in case of arbitrary P_{MO}.

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