Scenario-Free Analysis of Financial Stability with Interacting Contagion Channels

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Abstract

Currently financial stress test simulations that take into account multiple interacting contagion mechanisms are conditional on a specific, subjectively imposed stress-scenario. Eigenvalue-based approaches, in contrast, provide a scenario-independent measure of systemic stability, but only handle a single contagion mechanism. We develop an eigenvalue-based approach that gives the best of both worlds, allowing analysis of multiple, interacting contagion channels without the need to impose a subjective stress scenario. This allows us to demonstrate that the instability due to interacting channels can far exceed that of the sum of the individual channels acting alone. We derive an analytic formula in the limit of a large number of institutions that gives the instability threshold as a function of the relative size and intensity of contagion channels, providing valuable insights into financial stability whilst requiring very little data to be calibrated to real financial systems.

Keywords

Financial Stability, Systemic Risk, Interacting Contagion Channels, Financial Contagion, Multiplex Networks, Stress Test, Liquidity-Solvency Nexus

JEL Classification: G01, G17, G18, G21, G23, G28

1 Introduction

One of the revelations of the financial crisis was the importance of *systemic risk*, which is transmitted between institutions and amplified by their interactions. Risk control measures that are prudent for a single institution acting on its own may be counterproductive when many institutions act in unison [1, 2, 3, 4]. This problem is complicated by the fact that the financial system is heterogeneous, with different types of actors and different types of interactions [5, 6, 7]. This underscores the need to treat the financial system as a complex system to better understand the dynamics of risk transmission [8, 7]. Here we present a new method for understanding risk transmission when there are multiple types of risk contagion acting in tandem. Our method computes the linear stability of a financial system exposed to small shocks in a general setting. This makes it possible to estimate the stability of the financial system without having to impose subjective risk scenarios (in contrast to, for example, [9, 10]). As we show, taking all the channels of contagion and their interactions into account tends to make the system less stable, sometimes dramatically so.

The financial system can be thought of as a complex network consisting of different types of institutions such as banks, pension funds, hedge funds, money market funds and insurance companies [11, 8, 7]. The nodes of the network are the individual institutions and the links describe how they affect each another. The state of an institution corresponds to its *balance sheet*, i.e. its list of assets and liabilities. When an institution takes an action such as selling securities or withdrawing or defaulting on the payment of loans, this affects the balance sheets of other institutions. During a financial crisis institutions come under stress and these stress-transmission linkages become particularly strong [12]. This can give rise to instabilities and cause systemic risk. This raises a key challenge: How do we understand the collective stability of the financial system?

The mechanisms through which risk is transmitted from one institution to another are called *contagion channels* [13]. The methods that currently exist for analyzing multiple interacting channels depend on the generation of *scenarios*, consisting of specific sequences of external events that potentially threaten the financial system [13, 14, 15, 6, 7]. This has the obvious problem that scenarios are inherently subjective, causing debates about their realism [16, 7]. Scenarios are by their very nature not comprehensive [16] – the financial system might be stable under one set of scenarios and collapse under other unexpected and unanalyzed scenarios.

A better method explicitly models the financial network as a dynamical system, so that its stability can analyzed in terms of its eigenvalues. This has been done for contagion channels acting in isolation [17, 12], but so far has not been done for multiple channels operating at the same time. Given that the interaction of multiple channels can produce instabilities that far exceed their effect when acting alone [18, 13, 19], this is a dangerous state of affairs. Our key contribution here is to introduce a systematic method for analyzing a financial network with multi-contagion channel interactions as a dynamical system, thereby improving our ability to understand and monitor the stability of the financial system.

Here we analyze four principal contagion channels of the financial system, which we call funding contagion, overlapping portfolio contagion, counterparty risk contagion, and leverage targeting contagion. *Funding contagion* occurs when a borrowing institution depends on short-term loans to provide liquidity, and runs the risk that the lender might withdraw its loans [20, 18]. *Overlapping portfolio contagion* occurs when two institutions hold common securities. If either institution sells securities this drives prices down, lowering the securities' value [1, 18, 17, 21, 22, 6]. *Counterparty risk* occurs when a lender runs the risk that a borrower might default [23, 24, 25, 26, 27, 28, 12]. Finally, *leverage targeting contagion* occurs when an institution uses borrowed funds to purchase assets [1, 2, 4]. The ratio of debt to equity is called the *leverage* λ. It is common to target a particular leverage to control risk. If the value of assets drops, debt remains constant but equity decreases, so leverage increases. This forces the institution to pay off debt to maintain its target, thereby decreasing its liquidity.

The culmination of a severe financial crisis is usually the default of one or more institutions [29, 30]. An institution can default either because of insolvency or illiquidity. *Insolvency* occurs when asset values drop so that equity becomes negative, i.e. when the value of liabilities exceeds that of assets [14]. Default due to *illiquidity* occurs if an institution is unable to meet its payment obligations [6]. The two are not the same: An institution can default due to a liquidity shock even when it is solvent and vice versa. In fact, liquidity is the more direct threat; an institution may survive insolvency by maintaining liquidity and regaining solvency at a later date, but for our purposes here we neglect this possibility.

The stability of the financial system can be analyzed in terms of its response to external shocks, which can be classified either as liquidity shocks or valuation shocks according to the type of default they threaten to cause. The four contagion channels we study here interact through propagating both types of shocks and converting one into the other:

- *Propagation of liquidity shocks by funding contagion:* If institution A depends on a short-term loan from institution B, if B suddenly withdraws the loan to meet a liquidity shock it receives, then this causes a liquidity shock to A.

- *Propagation of valuation shocks by counterparty risk contagion:* If a valuation shock causes institution A's probability of default to rise, the risk-adjusted value of its debt to institution B falls, causing a valuation shock to B.

- *Conversion of liquidity shocks to valuation shocks by overlapping portfolio contagion:* If institution A suffers a liquidity shock it may be forced to sell securities. This depresses their price. If institution B also has a position in these securities it experiences a valuation shock.

- *Conversion of valuation shocks to liquidity shocks by leverage targeting:* If a valuation shock decreases institution A's equity, its leverage rises. To return to its target leverage, the institution must raise cash to pay off debt, causing a liquidity shock to itself. We neglect slower mechanisms to raise equity-capital, such as issuing new shares or retaining earnings.

We show how to describe the collective dynamics of these four interacting channels so that the stability of the financial system can be analyzed in a scenario-independent manner.

2 Results

The interactions of the four channels of contagion can be captured in a single matrix A which we call the *shock transition matrix*, as shown in Figure 1a. Assume discrete dynamics with time t. Let $\vec{x}_t^l$ be the N dimensional vector of liquidity shocks and $\vec{x}_t^v$ be the N dimensional vector of valuation shocks, where N is the number of financial institutions. The combined shock vector $\vec{x}_t$ of length $2N$ is

$$\vec{x}_t = \begin{bmatrix} \vec{x}_t^l \\ \vec{x}_t^v \end{bmatrix}. \tag{1}$$

The shock transition matrix A is the $2N \times 2N$ matrix that acts on the shock vector $\vec{x}_t$ according to

$$\vec{x}_{t+1} = A\vec{x}_t. \tag{2}$$

Given the distinction between the top and bottom half of $\vec{x}_t$, we decompose the shock transition matrix into its four quadrants:

$$A = \begin{bmatrix} A^{ll} & A^{vl} \\ A^{lv} & A^{vv} \end{bmatrix}, \tag{3}$$

where each of the components A^{ll}, A^{lv}, A^{vl} and A^{vv} are $N \times N$ matrices, so that equation (2) can be written in the form

$$\vec{x}_{t+1} = \begin{bmatrix} \vec{x}^l_{t+1} \\ \vec{x}^v_{t+1} \end{bmatrix} = \begin{bmatrix} A^{ll}\vec{x}^l_t + A^{lv}\vec{x}^l_t \\ A^{vl}\vec{x}^l_t + A^{vv}\vec{x}^v_t \end{bmatrix}. \tag{4}$$

Equation (4) makes explicit how the diagonal quadrant A^{ll} describes the propagation of liquidity shocks and A^{vv} the propagation of valuation shocks. The off-diagonal quadrant A^{lv} gives the conversion of liquidity to valuation shocks and A^{vl} the conversion of valuation to liquidity shocks. Figure 1b shows the corresponding contagion channels.

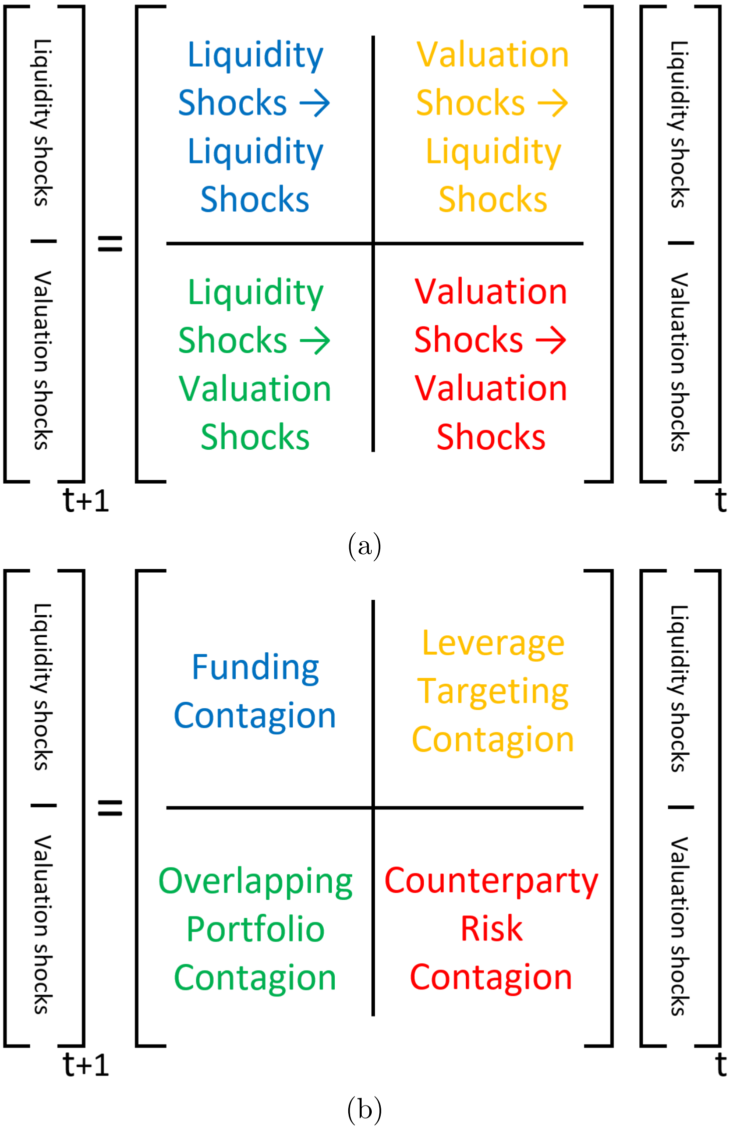


Figure 1: **Decomposition of shock dynamics.** The vector of shocks to the financial system can be written as a concatenation of the vector of liquidity shocks and vector of valuation shocks to each institution. The shock transition matrix maps the complete vector of shocks in one period to the vector of shocks in the next period. It can be decomposed into its four quadrants as shown, corresponding to the propagation and conversion of both shock types. Note the correspondence of the quadrants in (a) and (b): Funding contagion propagates liquidity shocks, counterparty risk propagates valuation shocks, overlapping portfolio contagion converts liquidity shocks to valuation shocks and leverage targeting converts valuation shocks to liquidity shocks.

The shock transition matrix A is the adjacency matrix of a weighted, directed, duplex network, where the nodes are institutions and the edges represent the transmission of shocks. Each institution is represented by a node in each layer. The top layer describes the propagation of liquidity shocks by funding contagion and is referred to as the *liquidity shock network*. The bottom layer describes the propagation of valuation shocks by counterparty risk contagion and is referred to as the *valuation shock network*. The edges between the two layers describe liquidity shocks transitioning to valuation shocks and vice versa, according to overlapping portfolio and leverage targeting contagion. This is illustrated in figure 2. Because we can express all contagion mechanisms in this two-layer system, in contrast to earlier methods, we do not need a separate layer for each contagion mechanism [18, 13, 19, 31, 32].

The shock transition matrix can be used to study the system's stability and resilience to shocks. Because all of its elements are non-negative the Perron-Frobenius theorem guarantees that it has a real eigenvalue whose absolute value is greater than or equal to that of the absolute value of the other eigenvalues [33]. This largest eigenvalue describes the systemic properties of the financial system: If the largest eigenvalue is greater than one, then shocks (that are not orthogonal to the corresponding eigenvector) are amplified without bound and we refer to the system as *inherently unstable*, and if the eigenvalue is smaller than one, they are damped. Hence, although no system is resilient to arbitrarily large shocks, an inherently unstable system is not even resilient to small shocks, as it amplifies them over time.

In equation (2), each contagion mechanism manifests itself in a single time step t. This implicitly assumes that all four contagion mechanisms act equally fast. However, we show in Supplementary Materials S.8 that the set of conditions under which the largest eigenvalue is equal to one is independent of this assumption.

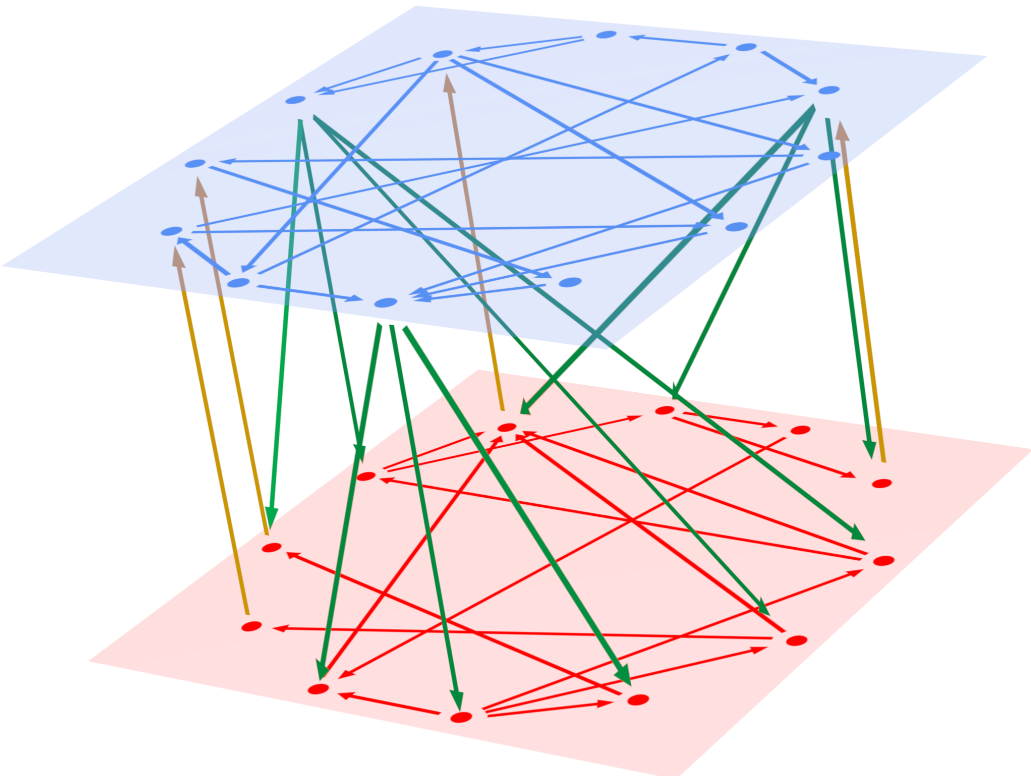


Figure 2: **The duplex network underlying the shock transition matrix A.** The nodes are institutions and the edges show the shock transmission between institutions. The top layer is the liquidity shock network (in blue) and the bottom layer the valuation shock network (in red). The green and yellow arrows represent interactions between the networks; the green arrow represents conversions of shocks from liquidity to valuation and yellow the conversion of shocks from valuation to liquidity. The shock transition matrix is a weighted adjacency matrix that describes both layers and their interactions at once.

2.1 Types of Institutions by Contagion Transmission

We classify institutions based on the contagion they transmit to study how a system's stability depends on its composition of different institutions. The response to a shock is generally size dependent. We focus on shocks that are small enough so that the response to shocks is fixed, which makes the dynamics approximately linear. The model can be extended to deal with larger shocks by updating the shock transition matrix as shocks propagate.

2.1.1 Responding to Liquidity Shocks

An institution may have multiple options available to respond to liquidity shocks. Here, we assume that each institution has a *pecking order* that specifies the order in which it sequentially uses these options as they are exhausted [13, 34]. For example, once an institution has fully sold its position in a given security, it may move on to selling another, less liquid, security. The assumption of a liquidity pecking order is fundamental to the Liquidity Coverage Ratio and Net Stable Funding Ration requirements [35, 36]. We focus on shocks that are sufficiently small that the option at the top of any institution's pecking order is not exhausted. In general the pecking order is institution-specific. Our methodology assumes that every institution has a pecking order, but any pecking order will work.
Here we assume that the pecking order minimizes liquidation costs [13, 34]. Any institution that holds sufficient cash on its balance sheet can absorb liquidity shocks without causing any contagion. We call such an institu-

tion a *liquidity sink*. Institutions that can easily access cash, for example by borrowing on the interbank market [37] or accessing central bank credit [38], can also act as liquidity sinks. (However, borrowing cash is not an option in response to a liquidity shock due to paying off debts to decrease leverage.) If an institution holds insufficient cash but has made short-term loans, it can raise cash by failing to roll over these loans. Finally it can liquidate securities; we assume this is done in order of liquidity. To summarize, cash is at the top of the pecking order, then withdrawal of short-term loans, then securities in order of liquidity.

2.1.2 Responding to Valuation Shocks

Valuation sinks are institutions with no leverage; they have no creditors to transmit contagion to and cannot have a leverage target, and therefore *absorb* valuation shocks. An example of a valuation sink is a pension fund which has no debt to the financial system.

We assume that each leveraged institution has a *leverage ceiling*, which reflects the maximal risk an institution is willing or allowed to take. If an institution is sufficiently close to its ceiling that a valuation shock would force it to delver then we say that it is *leverage targeting* [1, 22, 39, 6, 40]. In contrast, if the leverage is sufficiently below the ceiling (e.g. due to a leverage buffer, as proposed in recent regulation [41, 42]) we say that it is *passively leveraged*. We assume shocks that are sufficiently small that passively leveraged institutions do not transition to leverage targeting over time.

2.2 Contagion Equations

We now derive simple representative formulas for each contagion channel.

- *Funding contagion:* Suppose institution i extends a short-term loan of size S_{ij} to institution j, which is part of its short-term loan portfolio of size S_i. On receiving a liquidity shock x_i^l, assume institution i proportionately reduces the size of its short-term loans to each institution j to absorb the entire shock. This means that the liquidity shock that is transmitted to institution j is $A_{ji}^{ll} x_i^l$, where

$$A_{ji}^{ll} = \frac{S_{ij}}{S_i}. \tag{5}$$

- *Overlapping portfolio contagion:* Suppose institution i holds n_{si} shares of security s that is at the top of its pecking order, and experiences a liquidity shock x_i^l that causes it to sell $\Delta n_{si} = x_i^l / p_s$ shares, where p_s is the price of security s. Assume a price impact function of the form

$$\frac{\Delta p_s}{p_s} = \mu_s \frac{\Delta n_{si}}{n_s}, \tag{6}$$

  where n_s is the total number of shares of security s in circulation, and the price impact factor μ_s is a nondimensional constant of order one that is inversely proportional to the liquidity of security s. Setting $\mu_s = 1$ implies that selling n_s shares drives the price to zero. Under the assumption of linearity, $\mu_s = 1$ is an upper bound as the price cannot be negative. The resulting valuation shock to any institution j that holds n_{sj} shares of security s is $\Delta p_s n_{sj} = \mu_s x_i^l n_{sj} / n_s$, which implies

$$A^{vl}_{ji} = \mu_s \frac{n_{sj}}{n_s}. \tag{7}$$

Note that the diagonal component A^{vl}_{ii} is nonzero. We assume that institutions do not short securities, so we always have $n_{sj} \geq 0$.

- *Counterparty risk contagion:* Assume passively leveraged institution i has equity E_i and total debt D_i, so that its leverage is $\lambda_i = D_i/E_i$. When institution i experiences a valuation shock, its probability of default rises and the risk-adjusted value of its debt falls [12]. Institutions with more equity can withstand larger valuation shocks without becoming insolvent. Therefore, we assume that the fractional drop in the debt's value is proportional to the fractional loss in equity x^v_i/E_i. If institution i owes debt D_{ij} to institution j, then the valuation shock transmitted to institution j is $\delta_i x^v_i/E_i D_{ij}$, so

$$A^{vv}_{ji} = \delta_i \frac{1}{E_i} D_{ij} = \delta_i \lambda_i \frac{D_{ij}}{D_i}, \tag{8}$$

where the risk adjustment factor δ_i is a nondimensional constant of order one. Choosing $\delta_i = 1$ implies that a shock of size E_i (which causes bankruptcy) causes the full value of the debt to be lost and passed onto i's creditors as a valuation shock. Under the assumption of linearity, $\delta_i = 1$ is an upper bound as the loss cannot exceed the value of the debt. D includes short-term as well as long-term debt, so in general $D_{ij} \geq S_{ji}$.

- *Leverage targeting.* Suppose leverage targeting institution i maintains a leverage target λ_i. If it receives a valuation shock x_i^v it must pay off debt to return to its target. The amount by which it must reduce debt is $\lambda_i x_i^v$, so

$$A_{ii}^{lv} = \lambda_i. \tag{9}$$

  We assume that institution i's leverage targeting prevents the institution from transmitting counterparty risk contagion to its creditors. This is because the institution averts the risk associated with increased leverage by paying off its debts to keep its leverage constant.

In times of crisis, institutions sometimes hoard liquidity in response to liquidity shocks [20, 43]. Liquidity hoarding can be included in the funding (5) and overlapping portfolio (7) contagion equations by adding a hoarding term that captures the additional liquidity an institution hoards proportionally to the received liquidity shock. Here we make the simple assumption that liquidity hoarding is absent. This has the important implication that liquidity shocks are never amplified and the only source of shock-amplification in the financial system is the amplification of valuation shocks by leverage (see methods: *Aggregate Amplification*).

The contagion equations are summarized in Table 1. This set is not exhaustive; for example, information contagion is not included [44, 45]). These forms are chosen for simplicity – our basic methodology can apply to any contagion channels and does not depend on the details of the interaction terms.

Contagion Mechanism	Contagion Equation	Description
Funding Contagion	$A^{ll}_{ji} = \frac{S_{ij}}{S_i}$	Short-term lending withdrawal
Counterparty Risk Contagion	$A^{vv}_{ji} = \delta_i \lambda_i \frac{D_{ij}}{D_i}$	Probability of default increases due to lower valuations
Overlapping Portfolio Contagion	$A^{vl}_{ji} = \mu_s \frac{n_{sj}}{n_s}$	Price-impact of selling securities
Leverage Targeting Contagion	$A^{lv}_{ii} = \lambda_i$	Delevering requires raising liquidity

Table 1: Contagion Equations

2.3 Illustrative Example

We illustrate this approach by developing a simple example of a self-contained financial system that includes all four contagion mechanisms and both liquidity and valuation sinks. Consider four institutions, as summarized in Figure 3.

- Pension fund h has no debt and a cash surplus, making it both a valuation and a liquidity sink. It makes long-term loans L_{hi}, L_{hj} and L_{hk} to institutions i, j and k and has a position n_{sh} in security s.

- Bank i is passively leveraged. It has a position n_{si} in security s, which is at the top of its pecking order, and debt $D_{ih} = L_{hi}$ and $D_{ij} = S_{ji} + L_{ji}$ to institutions h and j.

- Bank j targets leverage λ_j. It makes short and long-term loans S_{ji}, L_{ji} and S_{jk}, L_{jk} to institutions i and k and has debt $D_{jh} = L_{hj}$ to institution h. The short-term loans are at the top of its pecking order.

- Bank k has a cash surplus, making it a liquidity sink, and maintains a leverage target λ_k. It has a position n_{sk} in security s, short-term debt $D_{kj} = S_{jk}$ and long-term debt $D_{kh} = L_{hk}$.

Balance sheet pension fund h	
Assets	Liabilities
Cash: surplus Long-term loans to institutions i, j and k	
Position in security s	Equity

(a)

Balance sheet bank i	
Assets	Liabilities
Cash: required minimum Position in security s	Debt to h and j
	Equity

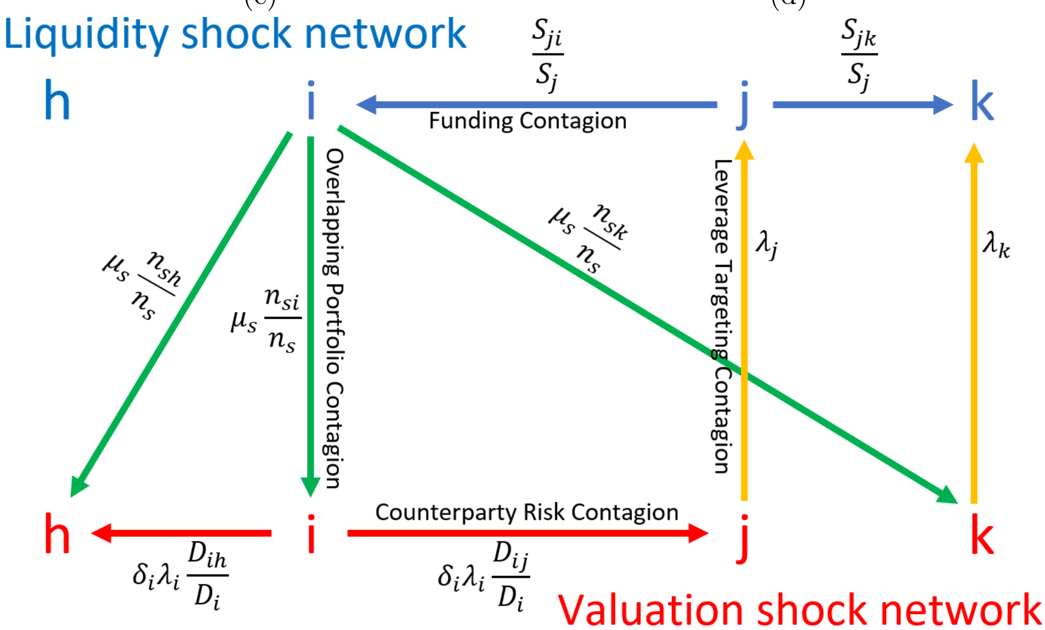
(b)

Balance sheet bank j	
Assets	Liabilities
Cash: required minimum Short and long-term loans to i and k	Debt to h
	Equity

(c)

Balance sheet bank k	
Assets	Liabilities
Cash: surplus Position in security s	Debt to h and j
	Equity

(d)



(e)

Figure 3: **A simple example illustrating the interaction of multiple channels of contagion.** Consider four institutions, h, i, j and k, whose balance sheets are given at the top of the figure. The edges denote their interactions, with expressions for the size of the interaction next to each edge. Institution h has no leverage and therefore transmits no shocks in response to a valuation shock, making it a valuation sink. Institution i is passively leveraged while j and k have a leverage target. Institution k has a cash surplus, which it uses to absorb liquidity shocks without transmitting shocks in response, making it a liquidity sink. **Blue edges:** When institution j receives a liquidity shock, it propagates it to i and k by proportionately reducing its short-term lending. **Green edges:** When institution i receives a liquidity shock, it sells part of its position in security s to raise liquidity. The resulting price-impact lowers the price of the security, causing a valuation shock to h, i and k. **Red edges:** When institution i receives a valuation shock its probability of default rises, transmitting a valuation shock to its creditors h and i by lowering the risk-adjusted value of its debt. **Yellow edges:** Institutions j and k have a leverage target and therefore use cash to pay off debt and delever in response to a valuation shock, causing a liquidity shock to themselves. (Because j and k keep their leverage constant, their probability of default is not increased by the valuation shock.)

The shock transition matrix of this system is

$$
\begin{array}{c|c}
 & \begin{array}{cccccccc} h^l & i^l & j^l & k^l & h^v & i^v & j^v & k^v \end{array} \\
\begin{array}{c} h^l \\ i^l \\ j^l \\ k^l \\ h^v \\ i^v \\ j^v \\ k^v \end{array} &
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{S_{ji}}{S_j} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda_j & 0 \\
0 & 0 & \frac{S_{jk}}{S_j} & 0 & 0 & 0 & 0 & \lambda_k \\
0 & \mu_s \frac{n_{sh}}{n_s} & 0 & 0 & 0 & \delta_i \lambda_i \frac{D_{ih}}{D_i} & 0 & 0 \\
0 & \mu_s \frac{n_{si}}{n_s} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \delta_i \lambda_i \frac{D_{ij}}{D_i} & 0 & 0 \\
0 & \mu_s \frac{n_{sk}}{n_s} & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} x^l_{t,h} \\ x^l_{t,i} \\ x^l_{t,j} \\ x^l_{t,k} \\ x^v_{t,h} \\ x^v_{t,i} \\ x^v_{t,j} \\ x^v_{t,k} \end{bmatrix}
=
\begin{bmatrix} x^l_{t+1,h} \\ x^l_{t+1,i} \\ x^l_{t+1,j} \\ x^l_{t+1,k} \\ x^v_{t+1,h} \\ x^v_{t+1,i} \\ x^v_{t+1,j} \\ x^v_{t+1,k} \end{bmatrix},
\end{array}
\quad (10)
$$

where $S_j = S_{ji} + S_{jk}$, $D_i = D_{ij} + D_{ik}$ and $n_s = n_{sh} + n_{si} + n_{sk}$.

To simplify the discussion we set the price-impact and risk adjustment factors to their upper bounds $\mu_s = \delta_i = 1$. In this case the largest eigenvalue of the shock transition matrix is equal to

$$\nu = \left(\frac{S_{ji} n_{si} \lambda_i D_{ij} \lambda_j}{S_j n_s D_i} \right)^{1/4}. \quad (11)$$

This is the product of each of the four contagion mechanisms, i.e.

$$\nu^4 = \frac{S_{ji}}{S_j} \times \frac{n_{si}}{n_s} \times \lambda_i \frac{D_{ij}}{D_i} \times \lambda_j.$$

The factors S_{ji}/S_j, n_{si}/n_s and D_{ij}/D_i are all less than or equal to one, and so exert a stabilizing force competing against the potentially destabilizing forces of the leverages λ_i and λ_j. If any of the four channels of contagion is removed the largest eigenvalue becomes zero and the system becomes unconditionally stable. The possibility for instability is caused by the interaction of all four channels, so analyzing each channel separately, as is normally done, gives an answer that is dangerously wrong.

To consider some plausible numbers, if $S_{ji}/S_j = D_{ij}/D_i = 1/3$, $n_{si}/n_s = 1/4$ and $\lambda_i = \lambda_j = 6$, then $\nu = 1$ and the system is at its margin of stability. Under the assumption that all leveraging institutions have the same leverage, we define the *critical leverage* as the leverage that makes the system unstable (in this case $\lambda_i = \lambda_j = 6$).

2.3.1 The stabilizing role of sinks

Shocks absorbed by sinks do not affect the largest eigenvalue. For example, the matrix entries in equation (10) that correspond to shock transmission to institutions h and k do not appear in equation 11. (See methods: *Omission of Sinks* for the general case.)

Nonetheless, from the contagion equations follows that the more short-term lending sinks receive, the more shocks are transmitted to sinks *and the less shocks are transmitted to non-sink institutions*. The same holds for how many securities sinks own and how much of institutions' debt sinks provide. The less shocks are transmitted to non-sink institutions, the less shocks propagate and the lower the largest eigenvalue becomes (see methods: *Omission of Sinks*).

This can be seen in the example by expanding the denominators in equation (11),

$$\nu = \left(\frac{S_{ji}}{S_{ji} + S_{jk}} \times \frac{n_{si}}{n_{sh} + n_{si} + n_{sk}} \times \lambda_i \frac{D_{ij}}{D_{ij} + D_{ih}} \times \lambda_j \right)^{1/4}. \qquad (12)$$

Sinks' short-term debt S_{jk}, securities holdings n_{sh} and n_{sk}, and lending D_{ih} all appear in the denominator only and exert a stabilizing force on the system.

Hence, *sinks stabilize the system by absorbing shocks that would otherwise have been transmitted to other institutions.*

2.4 Random Generation of Large Financial Systems

To demonstrate our method, we generate financial systems by randomly populating the balance sheets of N institutions with securities and loans. We study how the composition of the financial system affects the critical leverage by randomly assigning institutions to the types introduced previously, in the following proportions:

1. A fraction ϕ_l of institutions have sufficient cash to absorb shocks. We call ϕ_l the *fraction of liquidity sinks.*

2. A fraction ϕ_v of institutions have no leverage. We call ϕ_v the *fraction of valuation sinks.*

3. A fraction F of institutions provide short-term loans. We call F the *fraction of short-term lenders.*

4. A fraction Λ of *leveraged* institutions are leverage targeting. We call Λ the *fraction of leverage targeters.*

These constrain the random assignment of loans, as unleveraged institutions do not receive loans and only short-term lenders make short-term loans. Since an institution can either be a liquidity sink or not, provide short-term loans or not, and be unleveraged, passively leveraged or have a leverage target, this implies that there are $2 \times 2 \times 3 = 12$ different *types* of institutions.

Loans and securities are allocated to institutions that are chosen randomly and with uniform probability and replacement: For each security s out of

N^w distinct securities, we divide the total number of outstanding shares n_s into N^s blocks of n_s/N^s shares and assign each block to a randomly chosen institution. Each institution makes N^d loans, each to a randomly chosen leveraged institution, and all loans an institution *receives* are set equal in size. All loans made by short-term lenders are short-term and all other loans are long-term. In Supplementary Materials S.3, we show that our results are unaffected when part of the loans made by short-term lenders are long-term.

Once we choose the N^w distinct securities' market cap(italization)s and set all leveraged institutions' leverages equal to the critical leverage, the requirement that any institution's assets must equal the sum of its equity and debt fixes all remaining free parameters (see methods: *Balance Sheet Identity*). In Supplementary Materials S.2, we show that our results are not strongly affected when leverage varies across institutions.

2.4.1 Reduction to Representative Model

Under certain conditions, the shock transition matrix of the randomly generated financial system described above can be reduced to a 2×2 *representative matrix.* When $N, N^d/N, N^s/N \to \infty$, all institutions *of the same type* are indistinguishable; all institutions that transmit funding contagion, overlapping portfolio contagion and/or counterparty risk contagion do so identically and distribute the contagion homogeneously over all (other) institutions. Therefore, as shown in Supplementary Materials S.7, the system's dynamics are uniquely defined by the transmission of the aggregate liquidity shock $x^l_t = \sum_i x^l_{t,i}$ and aggregate valuation shock $x^v_t = \sum_i x^v_{t,i}$.

This allows us to reduce the full shock transition matrix to the 2×2 representative matrix $\hat{A}$ which describes the dynamics of the aggregate shocks,

$$\begin{bmatrix} x_{t+1}^{l} \\ x_{t+1}^{v} \end{bmatrix} = \left[\begin{array}{c||c} (1-\phi_l)F & \lambda(1-\phi_l)\Lambda \\ \hline\hline \mu(1-\phi_v)(1-F) & \delta\lambda(1-\phi_v)(1-\Lambda) \end{array}\right] \begin{bmatrix} x_{t}^{l} \\ x_{t}^{v} \end{bmatrix}, \tag{13}$$

where μ is the price impact factor of the most liquid security (all institutions have a position in this security when $N^s/N \to \infty$), and we have set $\delta_i = \delta$ for passively leveraged institutions and $\lambda_i = \lambda$ for all leveraged institutions. The representative matrix is analogous to a mean-field model. We call the system it describes the representative system.

We set $\mu = \delta = 1$ in the representative matrix, and solve its characteristic equation for the largest eigenvalue $\nu = 1$, which yields the *representative critical leverage*:

$$\hat{\lambda} = \frac{1-(1-\phi_l)F}{1-(1-\phi_l)F-\phi_l\Lambda}(1-\phi_v)^{-1}. \tag{14}$$

By having set the price-impact and risk-adjustment factors $\mu = \delta = 1$ to their upper bounds, we find the representative critical leverage that guarantees the system to not be inherently unstable, *regardless of the price-impact and risk-adjustment factors.*

2.4.2 Accuracy of Representative System

The representative critical leverage was derived from dense financial systems with many institutions. We now compare the representative critical leverage to critical leverages of randomly generated financial systems with varying N,N^d and N^s to evaluate its accuracy. As explained in the methods: *Calibration to Eurosystem*, calibrating to the Eurosystem yields the fraction of liquidity sinks $\phi_l = 0.75$, the fraction of valuation sinks $\phi_v = 0.2$, the fraction of short-term lenders $F = 0.5$ and the fraction of leverage targeters $\Lambda = 0.75$. We set the price impact and risk adjustment factors $\mu_s = \delta_i = 1$ and include $N^w = 10$ distinct securities whose markets caps are calibrated to the ten largest stocks on the Euronext exchange [46]. We generate 500 realizations of a random financial system with these parameters, according to the described procedure.

For each combination of N, N^d and N^s we plot the 50^{th} percentile (colored dot) and 15^{th} and 85^{th} percentiles (black bars) of the distribution of critical leverages sampled from 500 generated systems in figure 4. The representative critical leverage is plotted in red. We vary N from 10 to 100, N_d from 1 to 1000 and N_s from 10 to $10,000$.

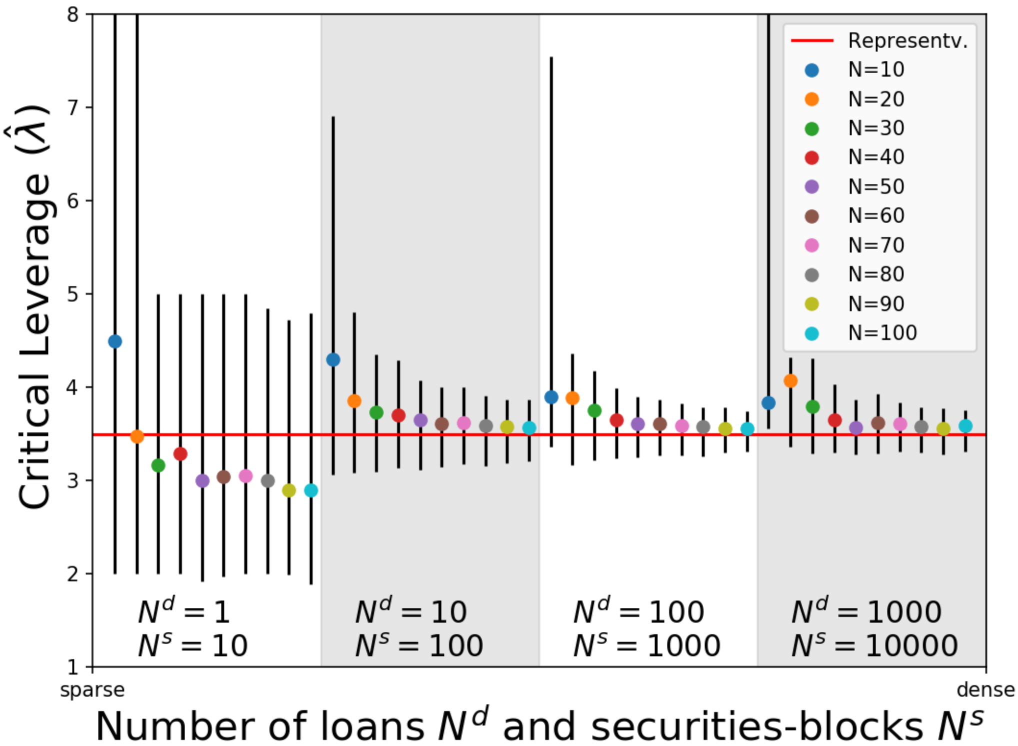


Figure 4: **Comparison of the representative model to randomly generated financial systems.** The plot compares the critical leverages of randomly generated financial systems to the representative critical leverage, the critical leverage of the representative system (the red line). We vary the number of loans N^d, the number of securities-blocks N^s and the number of institutions N. (For comparison, the Eurosystem has about 100 Significant Institutions [47].) For each combination of N, N^d and N^s, the colored dot shows the median and the black bars the 15^{th} and 85^{th} percentiles of the distribution of critical leverages obtained from generating 500 random systems. The figure shows that the simulated critical leverages quickly converge to the representative critical leverage as the number of institutions and density increases. As explained in the methods: *Calibration to Eurosystem*, the parameters $\phi_v = 0.2$, $\phi_l = 0.75$, $F = 0.5$ and $\Lambda = 0.75$ are calibrated to the Eurosystem. All leveraged institutions share the same leverage (but Supplementary Materials S.2 shows similar results when leverages vary across institutions). The price impact and risk adjusment factors $\mu_s = 1$ and $\delta_i = 1$ are set to their upper-bounds and $N^w = 10$ distinct securities are included, whose markets caps are calibrated to the ten largest stocks on the Euronext exchange [46].

Figure 4 shows that the representative critical leverage closely approximates the critical leverage as long as the system has at least roughly $N = 30$ institutions, each institution makes more than $N_d = 10$ loans, and there are at least $N_s = 100$ blocks of each security. Figure 4 also shows that systems with few institutions tend to be more stable than larger ones, as the median critical leverages show a downward trend as systems' numbers of institutions increase. As we show in Supplementary Materials S.4, the cause of this is that institutions cannot lend to or borrow from themselves. Furthermore, the medians in figure 4 also show that the sparsest systems ($N^d = 1$ and $N^s = 10$) are generally less stable; sparse networks typically include disconnected clusters, some of which will be unstable because of a heterogeneous distribution of sinks (see Supplementary Materials S.5).

The Eurosystem's 119 *significant institutions* as designated by the ECB have leverages ranging between 10 and 20 [47]. In contrast, in Figure 4 we estimate a critical leverage of roughly 3.5. We do not claim that the European financial system is unstable; these results only show the system to be unstable when the price-impact and risk-adjustment factors μ_s and δ_i are at their upper bounds of one. In practice, they are probably smaller. If we assume $\mu_s = \delta_i = 0.1$, for example, the representative critical leverage becomes $\hat{\lambda} = 35$ instead. Furthermore, the European system may include stabilizing forces our model fails to capture. For example, the exposures of the European system are likely far from random, and in particular hedging could raise the critical leverage considerably (see Supplementary Materials S.6). Finally, we have modeled the EU financial system as a closed system, whereas in reality it is highly interconnected with other parts of the global financial system. We chose to do so because the alternative, modeling it as an open system, would overestimate stability, because that would ignore feedback loops through the other parts of the global financial system.

2.4.3 Stability Overestimation

Most of the previous literature has studied the stability of the counterparty risk contagion channel in isolation [12]. Using the developed methodology, we show that omitting the interaction between contagion channels may lead to a severe overestimation of financial stability.

Stability in Isolation: Studies that analyze contagion channels in isolation omit the interaction between liquidity and valuation shocks. We derive the special conditions under which this is a reasonable approximation, which makes it clear that studying contagion channels in isolation almost always overestimates stability.

When all institutions are liquidity sinks ($\phi_l = 1$), the entire left half of the system's shock transition matrix is zero. Consequently, the largest eigenvalue of the shock transition matrix is given by the largest eigenvalue of its (bottom-right) counterparty risk contagion quadrant [33], i.e. the valuation shock network. Hence, the system is stable on the condition that the largest eigenvalue of the counterparty risk contagion quadrant does not exceed one. Depending on the context, we refer to this condition interchangeably as either the counterparty risk contagion channel or the valuation shock network being *stable in isolation*. Thus, in the unrealistic case that all institutions are liquidity sinks, the financial system is stable when the counterparty risk contagion channel is stable in isolation.

We demonstrate this on the representative system (13): Plugging $\phi_l = 1$ reduces the representative critical leverage (14) to

$$\hat{\lambda} = \frac{1}{1-\Lambda}(1-\phi_v)^{-1}, \tag{15}$$

which is the requirement that the counterparty risk contagion entry of the representative matrix equals one, i.e. that the the representative system's valuation shock network is stable in isolation. We refer to (15) as the *leverage stable in isolation* of the representative system. The representative critical leverage also reduces to the leverage stable in isolation when the fraction of short-term lenders $F = 1$, which is discussed in methods: *Aggregate Amplification*.

The stability of the valuation shock network in isolation, by definition, does not consider liquidity shocks and hence omits the feedback loop through which these liquidity shocks are converted back to valuation shocks. Therefore, the valuation shock network's stability in isolation is increased by the conversion of valuation to liquidity shocks (i.e. the absorption of valuation shocks through the creation of liquidity shocks). Equation (15) shows this: The higher the fraction of leverage targeters Λ, which corresponds to a higher conversion rate of valuation to liquidity shocks, the greater the stability in isolation.

The Misclassification Region: Studying the contagion channels in isolation can severely overestimate the stability of a financial system, especially when the interaction between contagion channels is strong. We quantify this using the representative system (13), but show in Supplementary Materials S.1 that the same holds for finite systems.

The representative critical leverage (14) and the leverage stable in isolation (15) diverge most strongly in the absence of liquidity sinks. When the fraction of liquidity sinks $\phi_l = 0$, liquidity shocks always return to the valuation shock network undamped. Therefore, any amplification of valuation shocks must be offset by damping by valuation sinks. Plugging $\phi_l = 0$ into the representative critical leverage (14) yields

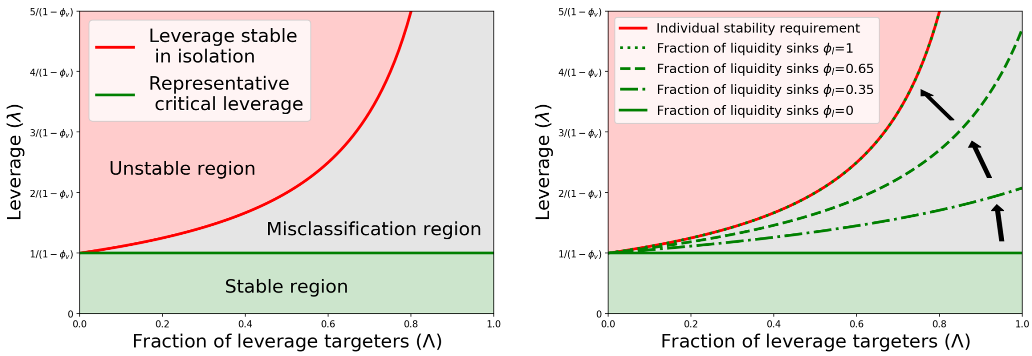
$$\hat{\lambda} = (1 - \phi_v)^{-1}, \tag{16}$$

which shows that when $\phi_l = 0$ the representative critical leverage depends only on the fraction of valuation sinks.

Figure 5a plots the leverage stable in isolation (15) and the representative critical leverage for the fraction of liquidity sinks $\phi_l = 0$ (16), as well as the *misclassification region* between the two. The misclassification region consists of leverages that seem stable when omitting the interactions between contagion channels, as these leverages are below the leverage stable in isolation, but that are actually destabilizing because they are above the critical leverage.

Figure 5b plots the representative critical leverage for various values of ϕ_l, showing that the misclassification region shrinks as the ϕ_l increases: As ϕ_l increases, the damping of liquidity shocks becomes stronger and the feedback loop between the liquidity and valuation shock networks weaker. (The effect of F and Λ on the critical leverage when $0 < \phi_l < 1$ is discussed in Supplementary Materials S.5.) This increases stability and, hence, the representative critical leverage, bringing it closer to the leverage stable in isolation.

However, the representative critical leverage (14) only coincides with the leverage stable in isolation (15) for the unrealistic case that either the fraction of liquidity sinks $\phi_l = 1$ or the fraction of short-term lenders $F = 1$, so a misclassification region always exists. Hence, omitting the interaction between contagion channels overestimates stability and carries the risk of classifying an unstable system as stable. For example, we estimate that the leverage stable in isolation overestimates the Eurosystem's critical leverage by almost 45%. Moreover, this overestimation soars to 300% when the sinks in the Eurosystem stop absorbing liquidity shocks, as may be the case during crises. ($\phi_l \to 0$ when sinks stop absorbing liquidity shocks, which reduces the representative critical leverage to equation (16).)



(a) (b)

Figure 5: **Overestimation of the stability of the financial system by omitting the interaction of channels of contagion.** The green line in (a) plots the critical leverage of the representative system when $\phi_l = 0$, i.e. when there are no liquidity sinks. The red curve in (a) plots the leverage stable in isolation, the critical leverage one finds when omitting the interactions between contagion channels. (Rather than fixing ϕ_v, we express the critical leverages in units of $(1-\phi_v)^{-1}$, as the critical leverages scale linearly in $(1-\phi_v)^{-1}$.) These are both plotted as a function of Λ, which describes the fraction of leverage targeting institutions. The region between the critical leverage and the leverage stable in isolation is the *misclassification region*. The misclassification region consists of destabilizing leverages that *seem* stable when considering channels in isolation. In (b) we vary ϕ_l, the fraction of institutions that are liquidity sinks, for the fraction of short-term lenders $F = 0.5$. (The critical leverages in (a) are independent of F.) The dashed lines in (b) show that the misclassification region shrinks as ϕ_l increases, but vanishes only when $\phi_l = 1$.

3 Discussion

Financial instability emanates from the endogenous amplification of shocks [48, 49, 50, 51], while resilient systems damp shocks. Therefore, to evaluate a system's inherent instability, methods to assess the financial system's tendency to amplify shocks are critical. We are the first to introduce a scenario-independent measure of the stability of multilayered financial systems. Our method describes the interactions of liquidity and valuation shocks. We capture multiple contagion mechanisms in a duplex network, consisting of a liquidity and a valuation shock layer. Although the model is simple, with only a few parameters, it is powerful enough to derive a wide range of insights about the stability of financial systems. Using this approach, we have shown in a scenario-independent framework that omitting the interaction between liquidity and valuation shocks overestimates stability, sometimes dramatically so. Since most studies focus on a single type of shocks [12], financial instabilities may be structurally underestimated.

One of the main challenges of the method is to calibrate it to real financial systems. To do this it is necessary to estimate whether and when institutions absorb liquidity shocks, to identify their leverage strategies and pecking orders, and measure the price-impact and risk-adjustment factors μ_s and δ_i. While this requires substantial further work, we think that it is feasible to do this, and it could result in a valuable new practical approach to evaluating financial stability. Other avenues of research include investigating how the stability of the system depends on the shock size, as larger shocks may exceed the top layers of pecking orders and the shock-absorption capacity of liquidity sinks.

4 Methods

4.1 Aggregate Amplification

A financial system may be unstable even when no institution ever transmits a shock to another institution that exceeds the shock it received. The instability of such a system is driven by *aggregate amplification*. We develop the concept of aggregate amplification to elucidate the claims we made that (1) liquidity shocks are never amplified (in the section *Contagion Equations*) and that (2) the critical leverage reduces to the individual stability requirement of the valuation shock network when the fraction of short-term lenders $F = 1$ (in the section *Stability Overestimation*).

We call the sum of the shocks an institution transmits to all institutions the aggregate transmitted shock. An institution's aggregate amplification gives the size of the aggregate transmitted shock relative to the received liquidity or valuation shock. Recall that an institution's column in the left half of the shock transition matrix gives the institution's shock transmission in response to a liquidity shock, so the sum of this column's entries gives the institution's aggregate amplification in response to liquidity shocks. Similarly, an institution's column-sum in the right half of the matrix gives the aggregate amplification in response to valuation shocks.

The importance of the aggregate amplification follows immediately from the Perron Fobrenius Theorem. The theorem implies that the largest eigenvalue of the shock transition matrix is bounded by its smallest and largest column-sums [33]. Consequently, the inherent instability of some financial systems can be evaluated without taking topological details into account, by considering only institutions' aggregate amplification:

- If all institutions' aggregate amplification in response to both liquidity and valuation shocks exceeds one, the system is inherently unstable.

- If no institution's aggregate amplification in response to either liquidity or valuation shocks exceeds one, the system is not inherently unstable.

If neither is true, the topology cannot be ignored and we need to calculate the largest eigenvalue to verify whether the system is inherently inherently unstable. This will likely be the case for most real financial systems. Nevertheless, even for such systems the aggregate amplification is a useful concept, as it allows us to identify specific destabilizing contagion channels based on that their aggregate amplification exceeds one.

For institution i, which withdraws short-term loans in response to a liquidity shock, the aggregate amplification is

$$\sum_{j=1}^{N} \frac{S_{ij}}{S_i} = 1. \tag{17}$$

For institution i, which sells securities in response to a liquidity shock, the aggregate amplification is

$$\sum_{j=1}^{N} \mu_s \frac{n_{sj}}{n_s} = \mu_s \leq 1. \tag{18}$$

For passively leveraged institution i, the aggregate amplification in response to a valuation shock is

$$\sum_{j=1}^{N} \delta_i \lambda_i \frac{D_{ij}}{D_i} = \delta_i \lambda_i \leq \lambda_i, \tag{19}$$

Finally, for leverage targeting institution i, the aggregate amplification in response to a valuation shock is simply

$$\lambda_i, \tag{20}$$

as the leverage targeting contagion is transmitted only to institution i itself.

Equations (17) and (18) show that the aggregate amplification in response to liquidity shocks never exceeds one, so the aggregate transmitted shock is never amplified relative to the received liquidity shock. This confirms claim (1). Consequently, the system cannot be inherently unstable when institutions' aggregate amplification in response to valuation shocks is smaller than or equal to one, which equations (19) and (20) show is the case when institutions' leverages do not exceed one.

Any column in the (upper-left) funding contagion quadrant of the shock transition matrix sums either to one, if the corresponding institution withdraws short-term loans to raise liquidity (17), or to zero otherwise. Therefore, the largest eigenvalue of the funding contagion quadrant never exceeds one. When the fraction of short-term lenders $F = 1$, the (bottom-left) overlapping portfolio contagion quadrant of the shock transition matrix is zero. Hence, the shock-transition matrix is upper block triangular and its eigenvalue spectrum is given by the union of its diagonal (funding and counterparty risk contagion) quadrants' eigenvalue spectra [33]. Consequently, the shock transition matrix' largest eigenvalue only exceeds one when the counterparty risk contagion quadrant's largest eigenvalue exceeds one. Hence, the critical leverage reduces to the leverage stable in isolation when $F = 1$, which confirms claim (2). The intuition here is that, because liquidity shocks are never converted to valuation shocks when institutions always have the option to withdraw short-term loans, the feedback-loop between the liquidity and valuation shock networks is incomplete and the system is stable when the valuation shock network is stable in isolation.

4.2 Omission of Sinks

In the Example, the transmission of shocks to sinks does not appear in the largest eigenvalue and, consequently, the sinks act as a stabilizing force. Here, we show that this is true in general. In fact, the rows of the shock transition matrix that correspond to liquidity shock transmission to liquidity sinks can be completely omitted from the matrix without affecting its largest eigenvalue. Moreover, the same holds for rows that correspond to valuation shock transmission to valuation sinks. We use the Example to demonstrate this, and explain that the general case follows straightforwardly.

Let us move the liquidity shocks received by liquidity sinks h and k, and valuation shocks received by valuation sink h to the end of the shock vector:

$$\hat{\vec{x}}_t = \left[x^l_{t,i}, x^l_{t,j}, x^v_{t,i}, x^v_{t,j}, x^v_{t,k}, x^l_{t,h}, x^l_{t,k}, x^v_{t,h}\right]^T, \tag{21}$$

This reorders the columns and rows of the corresponding shock transition matrix but leaves the largest eigenvalue unaffected:

$$\bar{A} = \begin{array}{c} \begin{array}{cccccccc} i^l & j^l & i^v & j^v & k^v & h^l & k^l & h^v \end{array} \\ \begin{bmatrix} 0 & \frac{S_{ji}}{S_j} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_j & 0 & 0 & 0 & 0 \\ \mu_s \frac{n_{si}}{n_s} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \delta_i \lambda_i \frac{D_{ij}}{D_i} & 0 & 0 & 0 & 0 & 0 \\ \mu_s \frac{n_{sk}}{n_s} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{S_{jk}}{S_j} & 0 & 0 & \lambda_k & 0 & 0 & 0 \\ \mu_s \frac{n_{sh}}{n_s} & 0 & \delta_i \lambda_i \frac{D_{ih}}{D_i} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ i^l \\ j^l \\ i^v \\ j^v \\ k^v \\ h^l \\ k^l \\ h^v \end{array} . \tag{22}$$

The three rightmost columns of the reordered shock transition matrix $\bar{A}$ are zero. Hence, the largest eigenvalue of $\bar{A}$ is equal to the largest eigenvalue of its upper-left 5×5 block, which we refer to as the *reduced shock transition matrix* [33].

The reduced shock transition matrix is obtained from the original shock transition matrix by removing the rows that correspond to liquidity and valuation shock transmission to, respectively, liquidity and valuation sinks, and removing the columns that correspond to the sinks' zero shock transmission in response to these shocks. As the reduced shock transition matrix's largest eigenvalue is identical to that of the original matrix, we conclude that omitting these rows and columns does not affect the largest eigenvalue.

This result generalizes to systems of any size and configuration: By definition, the shock transmission of liquidity and valuation sinks in response to, respectively, liquidity and valuation shocks is always zero. Therefore, we can always move the corresponding columns (of zeros) to the right of the shock transition matrix to obtain a reduced shock transition matrix whose eigenvalue is identical to that of the original matrix. Hence, the transmission of liquidity and valuation shocks to, respectively, liquidity and valuation sinks can always be omitted from the shock transition matrix without affecting the largest eigenvalue.

The reduced shock transition matrix only includes the transmission of shocks not absorbed by the receiving institution (i.e. a sink). From the contagion equations, we know that the shock transmission not absorbed by sinks decreases the more short-term debt liquidity sinks have, and the more securities and the more of institutions' debt valuation sinks own. Furthermore, in Supplementary Materials S.8, we show that the largest eigenvalue of a non-negative matrix is a monotonic increasing function of the matrix' entries (i.e. the corresponding network's edge-weights). Therefore, the reduced shock transition matrix' entries and, consequently, its largest eigenvalue, are monotonic decreasing in how much short-term debt liquidity sinks have, and how many securities and how much of institutions' debt valuation sinks own. As the shock transition matrix and reduced shock transition matrix share the same largest eigenvalue, this also holds for the shock transition matrix. Thus, sinks stabilize the system.

4.3 Balance Sheet Identity

The balance sheet requirement that any institution's assets must equal its debt plus equity must hold for each of the N institutions. This provides the constraints that fix the equities of the institutions in the randomly generated financial systems. To formulate the balance sheet requirement, we introduce some additional notation: For any institution i, let N_i^s denote the number of blocks of security s received, N_i^d the total number of loans received, and N_{ji}^d the number of loans received from institution j. Letting $C_s = p_s n_s$ denote the market cap of security s, institutions' equities are fixed endogenously by the requirement that institutions' assets (LHS), given by their securities and loan porfolios, equal debt plus equity (RHS):

$$\sum_{s=1}^{N^w} C_s \frac{N_i^s}{N^s} + \sum_{j=1}^{N^v} D_j \frac{N_{ji}^d}{N_j^d} = E_i \left(\lambda_i + 1\right), \tag{23}$$

where j runs over the $N^v = (1-\phi_v)N$ leveraged institutions, and $\lambda_i = 0$ for valuation sinks and $\lambda_i = \hat{\lambda}$ (the critical leverage) for all other institutions. As institutions' leverages fix their debts relative to their equities, equation (23) provides N constraints to solve for the N institutions' equities E_i. (The market caps C_s are fixed exogenously as part of the calibration.)

Note that the fraction $\frac{N_{ji}^d}{N_j^d}$ in equation (23) is undefined for any leveraged institution j that has not received any loans; by definition any leveraged institution must receive at least one loan. Therefore, for any financial system we generate, if a leveraged institution has not received any loans after each of the N institutions has made its N^d loans, the leveraged institution receives a loan from a randomly chosen institution.

4.4 Calibration to Eurosystem

The fraction of valuation sinks ϕ_v, the fraction of liquidity sinks ϕ_l, the fraction of short-term lenders F and the fraction of leverage targeters Λ are calibrated based on the types of financial institutions present in the EU financial system [52]. They are calibrated to the aggregate asset value of each type, rather than the number of institutions of each type, to take into account the strong heterogeneity in institutions' sizes (see Supplementary Materials S.5).

The fraction of valuation sinks ϕ_v is calculated as the fraction of the total assets of financial institutions held by pension funds and insurance companies, the fraction of short-term lenders F as the fraction of total assets held by institutions that are Monetary Financial Institutions (MFIs; banks, money market funds, etc.) and the fraction of leverage targeters Λ as the fraction of *leveraged* institutions' assets held by MFIs. The fraction of liquidity sinks ϕ_l is a general estimate that assumes that market liquidity is favourable and takes into account the high fraction of institutions that provide short-term lending.

$\phi_v = 0.2$ was rounded to the nearest multiple of a fifth to ensure that the number of valuation sinks is an integer when N is a multiple of five. Similarly, $F = 0.5$, $\Lambda = 0.75$, and $\phi_l = 0.75$ were rounded to the nearest multiple of a fourth: To further specify the described procedure of generating financial systems, we specifically designate a fraction ϕ_l and a fraction F of *leveraged* institutions as liquidity sinks and short-term lenders, rather than of all institutions (which is computationally more expensive). Hence, we do not designate whether valuation sinks are liquidity sinks and/or short-term lenders, as valuation sinks' response to liquidity shocks is irrelevant; valuation sinks do not receive liquidity shocks in our framework, as they neither have short-term debt, nor a leverage target. Therefore, as $\phi_v = 0.2$, rounding the other fractions to a multiple of a fourth ensures that the number of institutions of each type is an integer when N is a multiple of five.

Although the calibration suggests that the sets of leverage targeting institutions and short-term lenders coincide, we designate short-term lenders and leverage targeting institutions independently. Supplementary Materials S.5 discusses how stability is affected when the sets of leverage targeting institutions and short-term lenders coincide.

5 Table of Notation and Terminology

Notation	Description
E	Equity
D	Debt
λ	Leverage
$\hat{\lambda}$	Critical leverage
ν	Largest eigenvalue
A	Shock Transition Matrix
$\vec{x}$	Shock vector
$\vec{x}^l$	Liquidity shocks vector
$\vec{x}^v$	Valuation shocks vector
S	Short-term loan
L	Long-term loan
n_s	Total number of shares in security s in circulation
μ_s	Price-impact factor for security s
p_s	Price of security s
δ_i	Risk-adjustment factor for institution i
ϕ_l	Fraction of (institutions that are) liquidity sinks
ϕ_v	Fraction of (institutions that are) valuation sinks
F	Fraction of short-term lenders (institutions that provide short-term loans)
Λ	Fraction of leverage targeters (institutions that are leverage targeting)
C_s	Market capitalization of security s
N	Number of institutions
N^w	Number of distinct securities
N^s	Number of blocks of shares of security s
N^d	Number of debts (loans)
N_i^s	Number of blocks of security s received by institution i
N_i^d	Number of loans received by institution i
N_{ij}^d	Number of loans from institution i to institution j

Table 2: Notation and terminology

Supplementary Materials

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Author Contributions

Garbrand Wiersema is the lead author; he carried out most of the numerical analysis, designed the theoretical framework, and wrote the supplementary materials. The other authors contributed to all aspects of the article.

Competing Interests Statement

The authors report no competing interests.

Supplementary Materials

To:

Scenario-Free Analysis of Financial Stability with Interacting Contagion Channels

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Note: References to equations, figures, and table numbers without the "S." prefix refer to the main manuscript.

S.1 Stability Overestimation in Finite Financial Systems

In the section *Stability Overestimation*, we used the representative system to show that the leverage stable in isolation can severely overestimate stability. Here, we consider this overestimation for random systems, which are generated as described in the section *Random Generation of Large Financial Systems*. For each random system, we calculate the overestimation as the percentage increase from its critical leverage to its leverage stable in isolation. We generate 100 systems for each combination of the fraction of leverage targeters Λ and the fraction of liquidity sinks ϕ_l, and for each generated system we plot its overestimation as a dot in figure S.1. The fraction of short-term lenders $F = 0.5$, the fraction of valuation sinks $\phi_v = 0.2$, the number of institutions $N = 100$, the number of loans per institutions $N^d = 10$, and number of securities blocks $N^s = 100$. Figure S.1 also plots the overestimation based on the representative system as solid lines and shows that this approximates random systems' overestimation well.

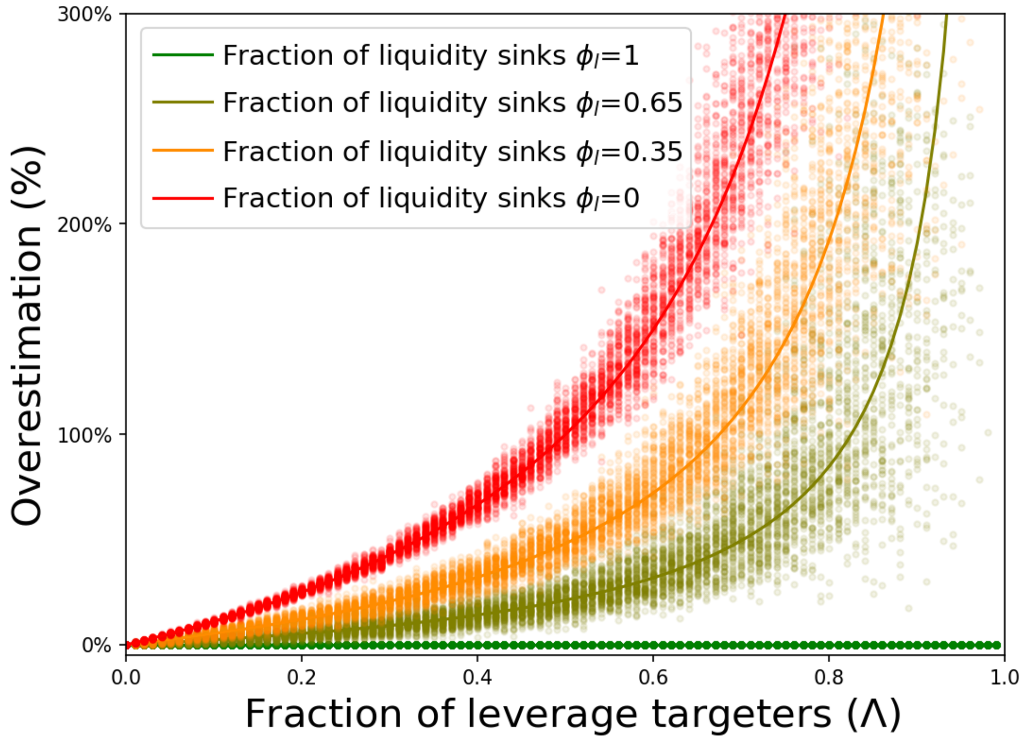


Figure S.1: **Overestimation of Critical Leverage by Individually Stable Leverage.** Comparison of the percentage increase from the critical leverage to the leverage stable in isolation of randomly generated systems (dots) and the representative system (solid lines). Fixed parameters: $F = 0.5$, $\phi_v = 0.2$, $\mu_s = 1$, $\delta_i = 1$, $N = 100$, $N^d = 10$, $N^s = 100$, and $N^w = 10$.

S.2 Critical Mean Leverage

Here, we consider the *critical mean leverage*, which is leveraged institutions' mean leverage for which the largest eigenvalue equals one. We generate random financial systems as described in the section *Random Generation of Large Financial Systems*, except that we draw leveraged institutions' leverages from a normal distribution with standard deviation σ and solve for the mean of this distribution for which the largest eigenvalue equals one, i.e. we solve for the critical mean leverage. Hence, when the standard deviation $\sigma = 0$, the critical mean leverage reduces to the critical leverage.

Figure S.2 plots the critical mean leverages of systems for a standard deviation $\sigma = 1$ and an otherwise identical calibration to the Eurosystem as in figure 4. The comparison of figures 4 and S.2 shows that the generated critical leverages and critical mean leverages show similar convergence to the representative critical leverage.

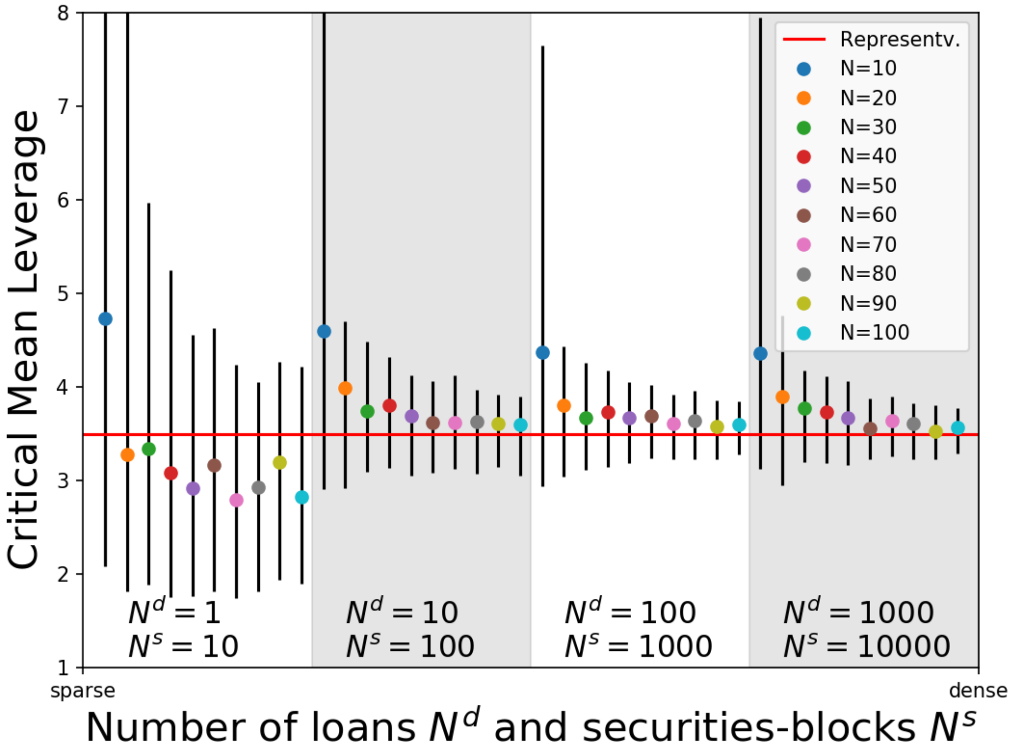


Figure S.2: **EU calibration with leverages varying across institutions.** Leveraged institutions' leverages are normally distributed with standard deviation $\sigma = 1$. We solve for the mean of the distribution for which the largest eigenvalue is equal to one, which we refer to as the critical mean leverage. Comparison with figure 4 shows that increasing $\sigma = 0$ to $\sigma = 1$ only increases variation somewhat (as shown by the increased percentile bars), but leaves the results otherwise unaffected. Fixed parameters: $\phi_v = 0.2$, $\phi_l = 0.75$, $F = 0.5$, $\Lambda = 0.75$, $\mu_s = 1$, $\delta_i = 1$, and $N^w = 10$.

S.3 Mixing Short- and Long-Term Lending

The systems is figure 4 were generated with short-term lenders providing only short-term loans. Here, we explore how the results are affected when part of the N^d loans provided by a short-term lender are long-term. Figure S.3 plots the critical leverages of financial systems in which half ($N^d/2$, rounded up) of the loans provided by each short-term lender are short-term, and the other half long-term. This leaves our results mostly unaffected, as can be seen from comparing figure S.3 to figure 4.

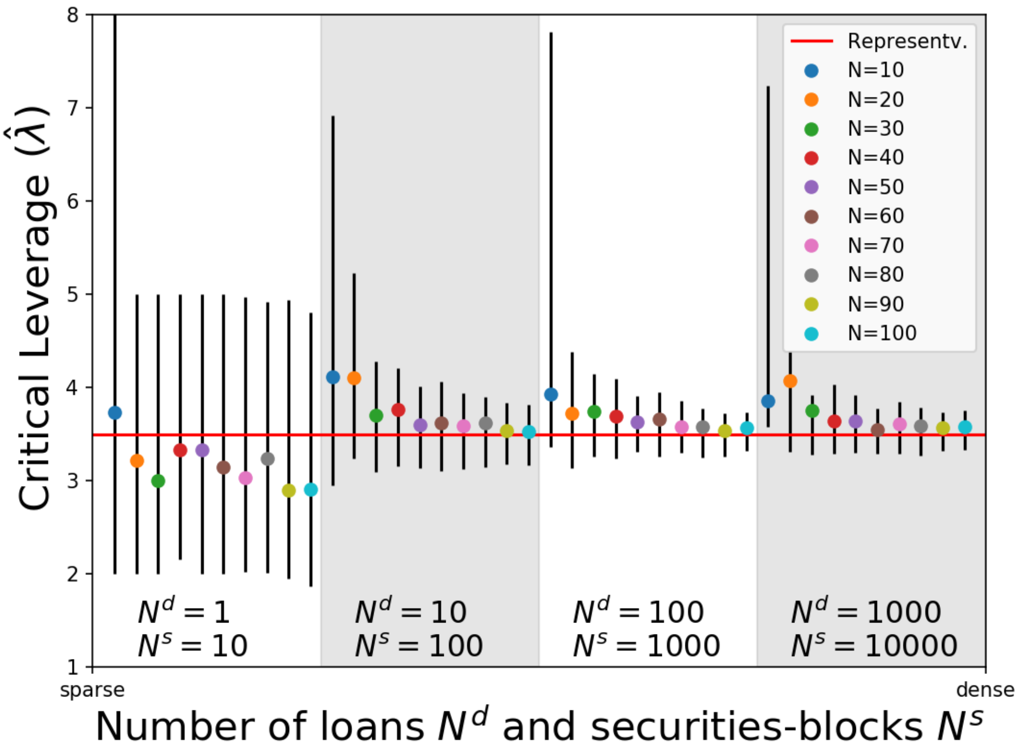


Figure S.3: **EU calibration with 50% of short-term lenders' loans being long-term.** The first 50% (rounded up) of loans made by a short-term lenders are short-term, and the remainder long-term. Comparison with figure 4 shows few differences. Fixed parameters: $\phi_v = 0.2$, $\phi_l = 0.75$, $F = 0.5$, $\Lambda = 0.75$, $\mu_s = 1$, $\sigma = 0$, $\delta_i = 1$, $h_i = 0$, $\omega = 0.5$, and $W = 100$.

S.4 Sensitivity to Price-Impact and Risk-Adjustment Factors

In figure 4, the price-impact factor μ and risk-adjustment factor δ were both set to their upper bound of one. Here, we investigate how the critical leverage is affected when these factors are set below their upper bounds.

To understand the price-impact and risk-adjustment factors' effect on the representative critical leverage, we calculate the representative critical leverage from the representative matrix (13) without setting the factors to their upper bounds of one, which yields

$$\hat{\lambda} = \frac{1 - (1 - \phi_l)\, F}{\mu\Lambda\,(1 - \phi_l)\,(1 - F) + \delta\,(1 - \Lambda)\,(1 - (1 - \phi_l)\, F)}(1 - \phi_v)^{-1}. \tag{S.1}$$

As the terms $1 - (1 - \phi_l)\, F$, $\Lambda\,(1 - \phi_l)\,(1 - F)$, $(1 - \Lambda)\,(1 - (1 - \phi_l)\, F)$ and $(1 - \phi_v)$ are all positive, the critical leverage increases as the price-impact and risk-adjustment factors μ and δ decrease from one to zero. This is to be expected, as a decrease in μ and δ corresponds to a decrease in the transmitted overlapping portfolio and counterparty risk contagion.

In figure S.4, we plot the representative critical leverage (S.1) and critical leverages of systems generated identically to those in figure 4, except that securities' price-impact factors range from $\mu = 0.5$ (for the most liquid security) to $\mu_s = 1$ (for the least liquid security). Recall that the price-impact factor $\mu = 0.5$ that appears in the representative critical leverage corresponds to the most liquid security; as N^s is raised to infinity, all institutions have a (liquidateable) position in the most liquid security. Comparing figures 4 and S.4 shows that, although the decrease in the price-impact factor μ has increased the critical leverages of the generated systems and the representative system, convergence to the representative critical leverage is qualitatively similar.

In figure S.5, we plot the representative critical leverage (S.1) with $\delta = 0.5$, and critical leverages of systems generated identically to those in figure 4, except that passively leveraged institutions' price-impact factors δ_i are normally distributed with a mean of 0.5 and a standard deviation of 0.1. Comparing figures 4 and S.5 shows that, although the decrease in the risk-adjustment factor has increased the critical leverages, convergence to the representative critical leverage remains qualitatively similar.

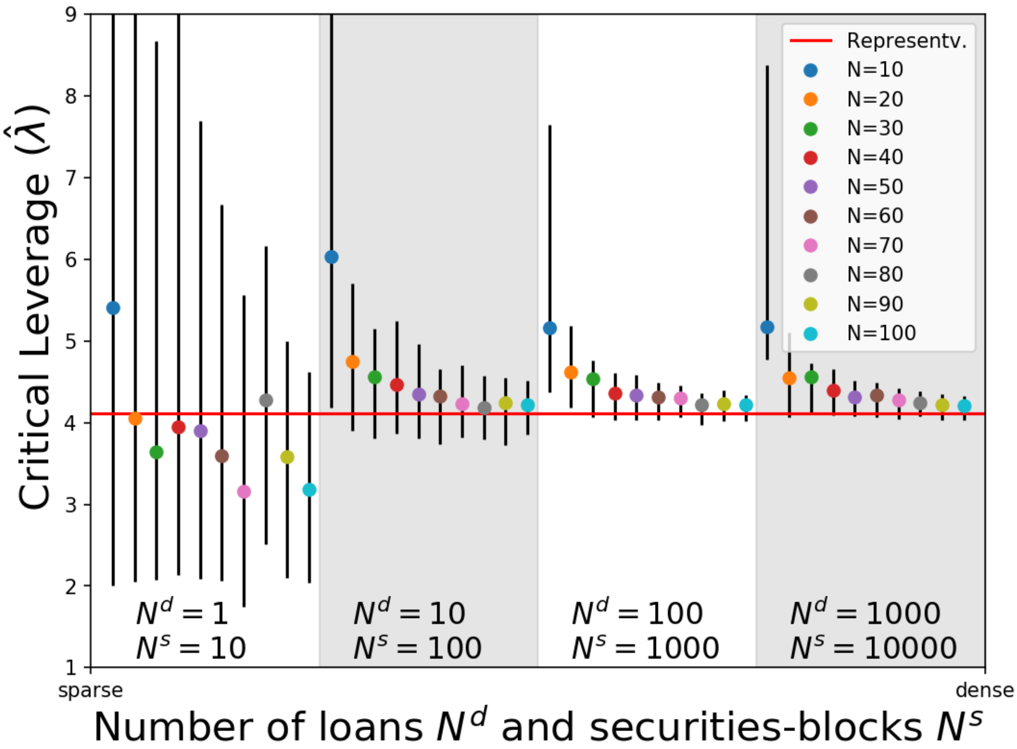


Figure S.4: **EU calibration with varying price-impact factor μ_s.** Price-impact factors μ_s vary from $\mu_s = .5$ for the most liquid security to $\mu_s = 1$ for the least liquid security. Compared to figure 4 where $\mu_s = 1$, varying μ_s below one increases the critical leverages, but the critical leverages nevertheless converge to the representative leverage with $\mu = 0.5$. Fixed parameters: $\phi_v = 0.2$, $\phi_l = 0.75$, $F = 0.5$, $\Lambda = 0.75$, $\sigma = 0$, $\delta_i = 1$, $h_i = 0$, $\omega = 1$, and $W = 100$.

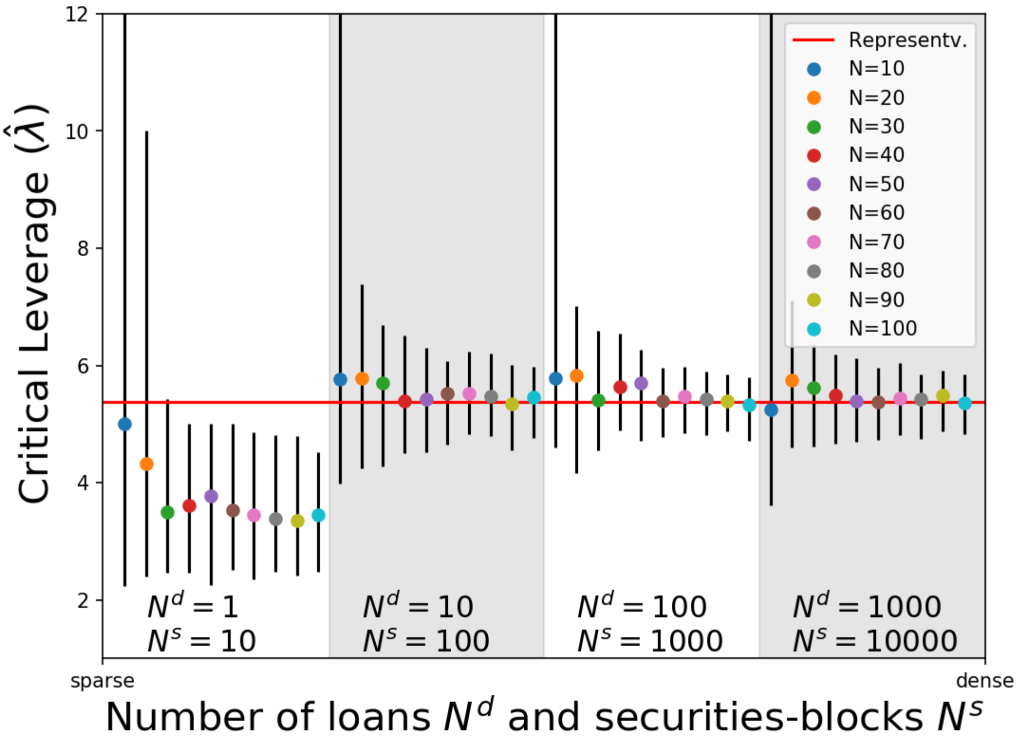


Figure S.5: **EU calibration with varying risk-adjustment factor δ_i.** Risk-adjustment factors δ_i are normally distributed with mean equal to 0.5 and standard deviation of 0.1. Compared to figure 4 where $\delta_i = 1$, varying δ_i below one increases the critical leverages, but the critical leverages nevertheless converge to the representative leverage with $\delta = 0.5$. Fixed parameters: $\phi_v = 0.2$, $\phi_l = 0.75$, $F = 0.5$, $\Lambda = 0.75$, $\sigma = 0$, $\mu_s = 1$, $h_i = 0$, $\omega = 1$, and $W = 100$.

S.5 Variation in Critical Leverages

Here, we discuss the causes of variation in systems' critical leverages. We distinguish between three different types of variation: Variation explained by the representative critical leverage, persistent biases that cause the medians of systems' critical leverages to differ from the representative critical leverage, and random fluctuations that cause systems' critical leverages to differ from the median (for a fixed set of parameters).

S.5.1 Variation Explained by the Representative Critical Leverage

As the representative critical leverage is a function only of the fractions of valuation and liquidity sinks ϕ_v and ϕ_l and fractions of short-term lenders and leverage targeters F and Λ, it can explain variation driven by these parameters but not any others. The effect of some of these fraction on the representative critical leverage was already briefly discussed in section *Stability Overestimation*, but here we provide a more detailed explanation.

Valuation Sinks Fraction:

From the derivation in Appendix S.7 follows that we can distinguish between passively leveraged institutions' leverage $\lambda_i = \lambda^\rho$ and leverage targeting institutions' leverage $\lambda_i = \lambda^\tau$ to obtain the representative matrix

$$\hat{A} = \left[\begin{array}{c||c} (1-\phi_l)F & \lambda^\tau(1-\phi_l)\Lambda \\ \hline\hline (1-\phi_v)(1-F) & \lambda^\rho(1-\phi_v)(1-\Lambda) \end{array}\right]. \tag{S.2}$$

From the representative matrix in equation (S.2) follows immediately that valuation sinks can directly offset the amplification by passively leveraged institutions' leverages (in the representative system); in the counterparty risk entry of the matrix, the leverage and valuation sinks only appear as the product $\lambda^\rho(1-\phi_v)$. Thus, when $\lambda^\rho \leq 1/(1-\phi_v)$, their product does not exceed one and counterparty risk contagion does not cause aggregate amplification in the representative system (see methods: *Aggregate Amplification*).

Furthermore, the representative critical leverage (13), the leverage which can be attained by both passively leveraged and leverage targeting institutions without making the system inherently unstable, can be written as

$$(1-\phi_v)\hat{\lambda} = \frac{1-(1-\phi_l)F}{1-(1-\phi_l)F-\phi_l\Lambda}. \tag{S.3}$$

As the RHS of equation S.3 is always greater than or equal to one, the system is never unstable whenever all leveraged institutions (both passively leveraged and leverage targeting) have a leverage $\hat{\lambda} \leq 1/(1-\phi_v)$. Thus, amplification of leverage targeting institutions' leverages is also offset by valuation sinks when their leverages do not exceed $1/(1-\phi_v)$. The reason for this is as follows:

From Appendix S.8, we know that whether the largest eigenvalue is equal to one depends only the edge-weight-products of cycles in the duplex contagion network described by the shock transition matrix. Any cycle must transition from the liquidity shock network to the valuation shock network (through overlapping portfolio contagion) and transition from the valaution shock network to the liquidity shock network (through leverage targeting contagion) an equal number of times, in order to return to the starting node of the cycle. As such, each cycle's edge-weight-product must include an equal number of overlapping portfolio and leverage targeting contagion terms. For the representative system specifically, this means that the edge-weight-product includes and equal number of $(1-\phi_v)$ terms and leverage targeting institutions' leverages λ^τ. Hence, only their product $(1-\phi_v)\lambda^\tau$ is relevant for stability.

As only the product $(1-\phi_v)\hat{\lambda}$ affects the stability of the representative system, expressing the representative critical leverage $\hat{\lambda}$ in units of $(1-\phi_v)^{-1}$, as was done in figure 5, allows us to focus on how stability is affected by the fractions of liquidity sinks, short-term lenders and leverage targeters ϕ_l, F and Λ. We discuss how these fractions affect stability now.

Liquidity sinks, short-term lending and leverage targeting fractions:

The effects of the fractions of liquidity sinks, short-term lenders and leverage targeters ϕ_l, F and Λ on the critical leverage (and, hence, stability) are interdependent and therefore cannot be considered in isolation. When the fraction of liquidity sinks $\phi_l = 0$, the fractions of short-term lenders and leverage targeters F and Λ do not affect the critical leverage (16); the rates at which valuation shocks transition to liquidity shocks and vice versa do not matter, as valuation shocks that transition to liquidity shocks always transition back undamped eventually.

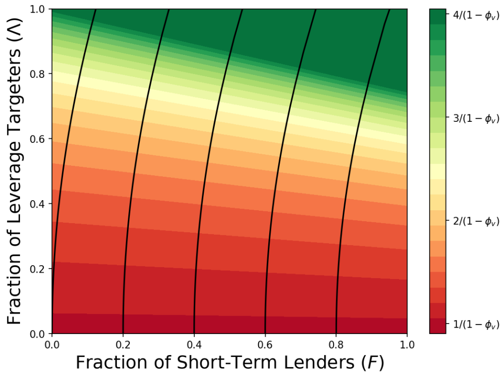
On the other hand, when the fraction of liquidity sinks $\phi_l = 1$ the representative critical leverage is independent of the fraction of short-term lenders, and when the fraction of short-term lenders $F = 1$, the representative critical leverage is independent of the fraction of liquidity sinks (15); in either case, valuations shocks that transition to liquidity shocks never transition back to valuation shocks, so the other fraction does not matter. (Of course the rate at which valuation shocks transition to liquidity shocks, given by Λ, *does* affect stability.)

Moreover, when the fraction of leverage targeters $\Lambda = 0$, no valuation shocks transition to liquidity shocks, so the fractions of liquidity sinks and short-term lenders are irrelevant and the representative critical leverage in units of $(1 - \phi_v)^{-1}$ cannot exceed one.

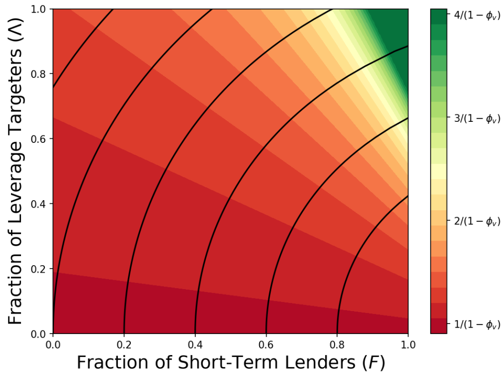
In figures S.6a and S.6b, we plot contours of the representative critical leverage (in units of $(1 - \phi_v)^{-1}$) as a function of the fractions of short-term lenders and leverage targeting F and Λ, for a low respectively high fraction of liquidity sinks ϕ_l. The black curves plot trajectories that follow the gradient at each point. The plots show that the representative critical leverage generally increases more strongly in the fraction of leverage targeters than the fraction of short-term lenders, but this effect is most pronounced when the fraction of liquidity sinks ϕ_l is high.

Figure S.7 plots contours of the *critical fraction of liquidity sinks* $\hat{\phi}_l$; the fraction of liquidity sinks for which the largest eigenvalue equals one in the representative system. The figure plots the critical fraction of liquidity sinks as a function of the fractions of short-term lenders and leverage targeters F and Λ, for $\lambda = 1.5/(1 - \phi_v)$. We see that the critical fraction of liquidity sinks decreases more strongly in the fraction of leverage targeters than the fraction of short-term lenders, except in the region where the critical fraction of liquidity sinks is already very small (i.e. where the system is already very stable). Different values of λ (in units of $(1-\phi_v)^{-1}$) produce qualitatively similar results. Lastly, note that the plot starts at $\Lambda = 1/3$, as for $\Lambda < 1/3$ and $\lambda = 1.5/(1-\phi_v)$ the largest eigenvalue always exceeds one, regardless of the fraction of liquidity sinks.

Hence, from figures S.6a-S.7 follows that the fraction of leverage targeters Λ generally affects stability more strongly than than the fraction of short-term lenders F. The reason for this is that even when $F = 0$ there is damping of liquidity shocks, whereas when $\Lambda = 0$, no shocks transition to the liquidity shock network (where they would be damped).



(a)



(b)

Figure S.6: **Contour plot of the critical leverage for high and low damping.** We plot the critical leverage for high damping ($\phi_l = .75$) in (a) and low damping ($\phi_l = .25$) in (b).

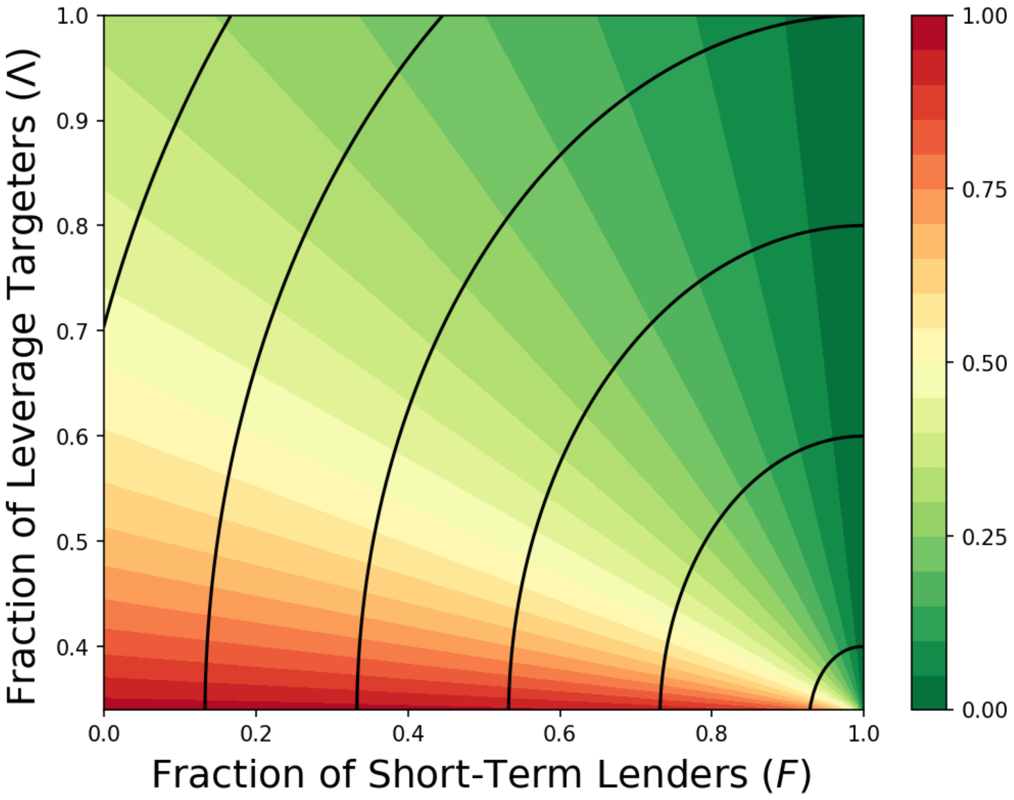


Figure S.7: **Contour plot of the critical fraction of liquidity sinks for** $(1-\phi_v)\hat{\lambda} = 1.5$**.**

S.5.2 Persistent Biases

The medians in figure 4 show two trends: the sparsest systems are less stable than the denser ones, and systems with few institutions are more stable than those with many. We discuss what causes these trends in this order.

The sparser a financial network is, the more likely it is that it consists of disconnected clusters, and, equivalently, that its shock transition matrix can be made block-diagonal (by reordering its rows and columns – see Appendix S.8), with the blocks corresponding to the clusters. The largest eigenvalue of a block-diagonal matrix is given by the largest of the blocks' eigenvalues. Hence, the system's stability is limited by its least stable cluster. Consequently, when sinks are distributed heterogeneously over the clusters, some clusters will be relatively unstable and the critical leverage low. When the network is denser, stable and unstable regions of the network are more likely to be connected, which offsets local instabilities such that the critical leverage is higher.

The relative stability of systems with few institutions, compared to those with many, is caused by the fact that an institution cannot lend to or borrow from itself. This is demonstrated by figure S.8, which shows that in generated systems where institutions are allow to lend to themselves, systems with few institutions are equally stable as those with many.

Because an institution cannot lend to itself, the institution is less likely to lend to or borrow from an institution of the same type than implied by the corresponding fraction of institutions of this type. For example, although a fraction Λ of leveraged institutions (institutions that receive loans) are leverage targeters, any loan a leverage targeter makes has a probability

$$(\Lambda N - 1)/(N - 1) < \Lambda \tag{S.4}$$

to be received by a leverage targeter. Consequently, passively leveraged institutions are less likely to borrow from passively leveraged institutions, and counterparty risk contagion is more likely to be transmitted to leverage targeters, than the fraction of leverage targeters Λ implies. This *stabilizes* the system. (Leverage targeters are stabilizing relative to passively leveraged institutions.) This effect is most pronounced for systems with few institutions, as the probability in equation (S.4) converges to Λ as $N \to \infty$.

Similarly, short-term lenders are less likely to lend to short-term lenders than the fraction of short-term lenders F implies. Consequently, funding contagion is more likely to be transmitted to institutions that sell securities to raise liquidity, which *destabilizes* the system. (Institutions that sell securities are destabilizing relative to those that withdraw short-term loans to raise liquidity.) Again, the effect is more pronounced for systems with few institutions.

The two effects described above can be interpreted as changing the "effective" fractions of leverage targeters and short-term lenders in systems with few institutions. We know from figures S.6a and S.6b that the critical leverage (and, hence, stability) is generally more sensitive to changes in the fraction of leverage targeters, than to changes in the fraction of short-term lenders. Hence, the net effect of the two effects described above is generally stabilizing.

Furthermore, in systems with few institutions, any short-term loan made by a non-sink institution is also more likely to be received by a liquidity sink than the fraction of liquidity sinks ϕ_l implies. Consequently, funding contagion is more likely to be transmitted to liquidity sinks and subsequently absorbed. This further stabilizes systems with few institutions.

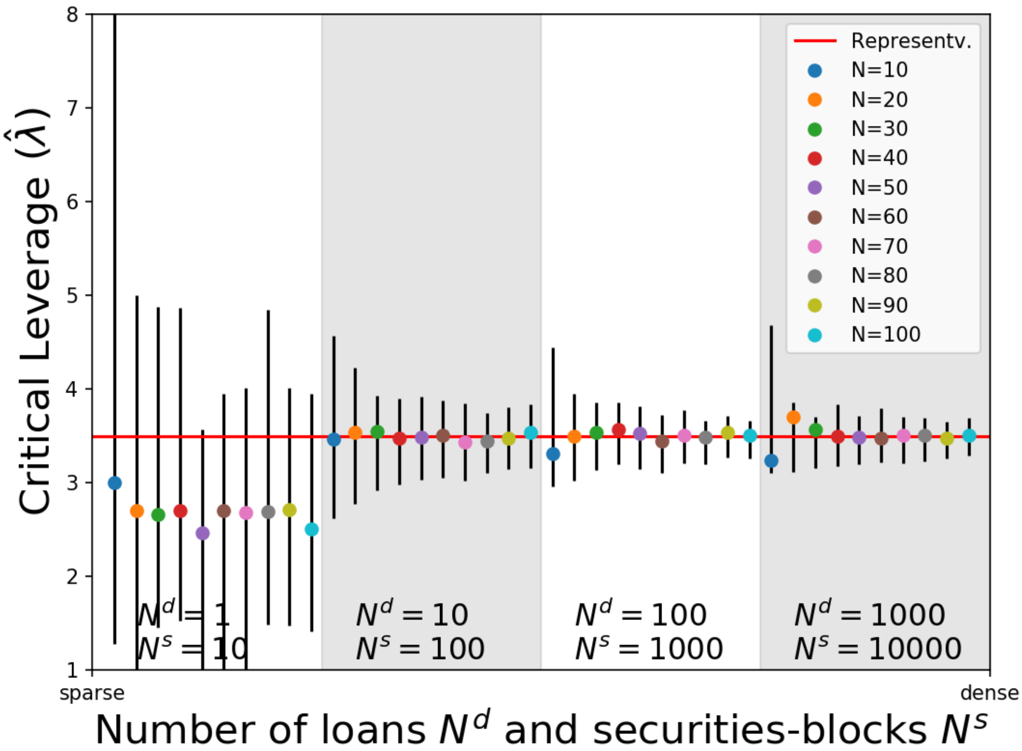


Figure S.8: **EU calibration with institutions being allowed to lend to themselves.** Fixed parameters: $\phi_v = 0.2$, $\phi_l = 0.75$, $F = 0.5$, $\Lambda = 0.75$, $\sigma = 0$, $\mu_s = 1$, $\delta_i = 1$, $h_i = 0$, $\omega = 1$, and $W = 100$.

S.5.3 Random Fluctuations:

Here, we explain how the random allocation of exposures and designation of institutions' types cause variation in the critical leverages of systems generated according to identical parameters.

Allocation of Exposures: Because funding, overlapping portfolio and counterparty risk contagion are distributed proportional to exposures, institutions that are allocated more exposures, and therefore have larger balance sheets, receive and transmit more contagion. Consequently, if the larger institutions are stabilizing relative to the smaller institutions, the system is more stable and the critical leverage higher, and vice versa. This suggest that for financial systems whose institutions are strongly heterogeneous in size, calibrating the representative critical leverage to institutions' aggregate asset size rather than their numbers may be more accurate.

Correlations Between Pecking Orders and Leverage Strategies: Correlations between institutions' leverage strategies and pecking orders affect stability. When more than a fraction ϕ_l of leverage targeting institutions are liquidity sinks, the probability that leverage targeting contagion is absorbed by liquidity sinks is increased. This increases stability relative to the case that the fraction of leverage targeting institutions that are liquidity sinks is exactly equal to ϕ_l (i.e. no correlation between liquidity sinks and leverage strategies). Of course, the reverse is also true; when less than a fraction ϕ_l of leverage targeting institutions are liquidity sinks, stability is decreased.

Furthermore, correlation whether institutions are short-term lenders and the institutions' leverage strategies also affects stability. The cause of this is the asymmetry between how overlapping portfolio and leverage targeting contagion are distributed; whereas the transmission of overlapping portfolio contagion is diffuse, leverage targeting contagion is only transmitted to the transmitting institution itself. Therefore, when the fraction of leverage targeting institutions that are short-term lenders is greater than F, leverage targeting contagion is more likely to subsequently cause funding contagion rather than overlapping portfolio contagion. This increases stability and the reverse correlation decreases stability. (Funding contagion is stabilizing relative to overlapping portfolio contagion.)

Moreover, when the fraction of leverage targeting institutions that are short-term lenders is greater than F, the fraction of passively leveraged institutions that are short-term lenders is smaller than F. This increases the probability that overlapping portfolio contagion is transmitted to passively leveraged institutions (rather than leverage targeting institutions), as overlapping portfolio contagion is always partially transmitted to the institutions that itself sold the securities. This decreases stability. However, the effect is marginal; most of the overlapping portfolio contagion transmitted to institutions other than the institution that itself sold the securities (and these institutions' leverage strategies are unaffected by any correlations with their pecking orders). Consequently, the net effect of a (positive) correlation between whether institutions are leverage targeting and whether they are short-term lenders is stabilizing. This is demonstrated by figure S.9, which shows that the critical leverage (and, hence, stability) is increased when the sets of short-term lenders and leverage targeting institutions coincide.

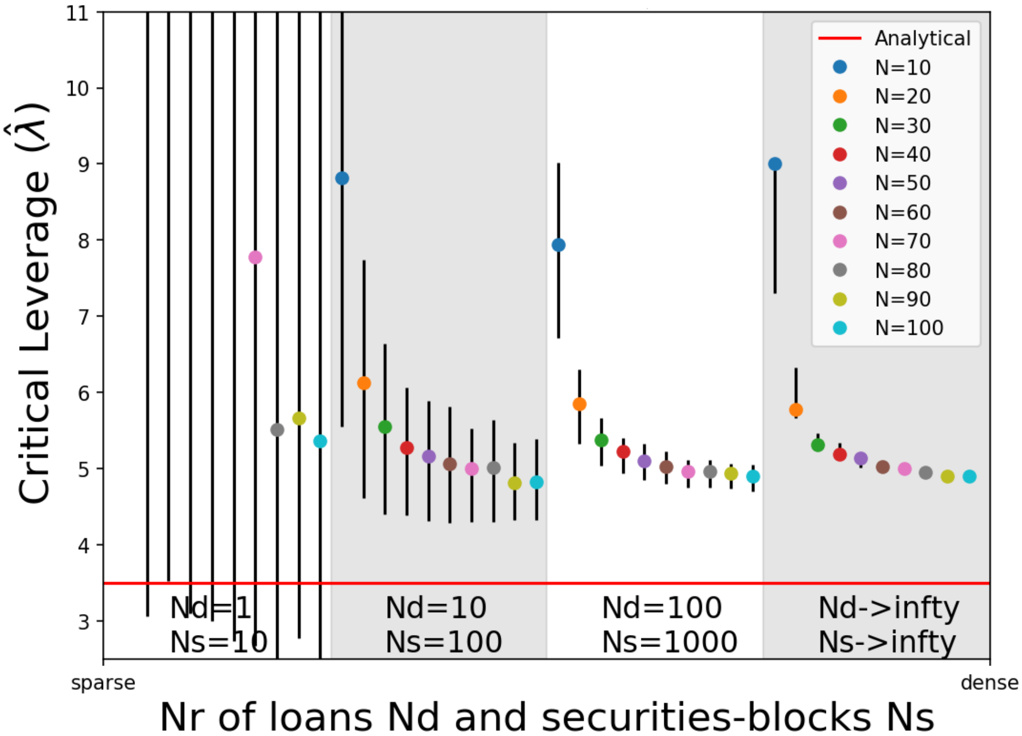


Figure S.9: **EU calibration with the sets of short-term lending and leverage targeting institutions coinciding.** Fixed parameters: $\phi_v = 0.2$, $\phi_l = 0.75$, $F = 0.5$, $\Lambda = 0.75$, $\sigma = 0$, $\mu_s = 1$, $\delta_i = 1$, $h_i = 0$, $\omega = 1$, and $W = 100$.

S.6 The Effects of Hedging

Institutions can hedge their positions in securities by taking short-positions in securities whose price movements are correlated to the security in which the institution is long. However, the system as a whole must be long in any security, so hedging does not affect to total amount of overlapping portfolio contagion that results from the sale of a security. Hence, all hedging can achieve is redistributing the overlapping portfolio contagion. As being short in a security constitutes a liability, valuation sinks can only be long in securities. Therefore, hedging has the potential to stabilize the system by shifting overlapping portfolio contagion from leveraged institutions (which amplify shocks), to valuation sinks (which absorb shocks). In the unrealistic case that hedging shifts *all* overlapping portfolio contagion to valuation sinks, all overlapping portfolio contagion is absorbed. Consequently, the critical leverage reduces to the leverage stable in isolation. From section *Stability Overestimation*, we know that this increases the critical leverage by about 45% in the Eurosystem (relative to the case without hedging).

S.7 Derivation of Representative Agent Model

Here, we show that the shock transition matrix reduces to a 2×2 representative matrix when passively leveraged institutions have the same risk-adjustment factor $\delta_i = \delta$, all leveraged institutions have the same leverage $\lambda_i = \lambda$, and $N^s/N, N^d/N, N \to \infty$. We denote the number of leveraged institutions as $N^v = (1-\phi_v)N$ and the number of non-sink institutions as $N^l = (1-\phi_l)N^v$. (As explained in the methods: *Calibration to Eurosystem*, we specifically designate a fraction ϕ_l of *leveraged* institutions as liquidity sinks, rather than of all institutions.)

We simplify the notation of the shock transition matrix in three steps:

1. Following procedure explained in the methods: *Omission of Sinks*, we remove the rows that correspond to liquidity and valuation shock transmission to, respectively, liquidity and valuation sinks, and the columns that correspond to the sinks' zero shock transmission in response.

2. As valuation sinks cannot receive liquidity shocks (they have no leverage target and no short-term debt), the rows that correspond to liquidity shocks transmission to valuation sinks are zero. We therefore also remove the rows that correspond to liquidity shock transmission to valuation sinks and columns that correspond to valuation sinks' shock transmission in response to liquidity shocks. We do this using a procedure similar to the one used in the previous step, but by creating a band of zero rows at the bottom of the matrix, rather than a band of zero columns on the right.

3. We move the valuation shocks x_i^v received by liquidity sinks i to the bottom of the shock vector, such that the corresponding rows and columns in the shock transition matrix are moved to the bottom and right end, respectively.

We refer to resulting matrix as the *simplified* shock transition matrix. The liquidity shock vector $\vec{x}^l$ is of length N^l and the valuation shock vector $\vec{x}^v$ is of length N^v. Hence, the dimensions of the simplified shock transition matrix' funding contagion quadrant are $N^l \times N^l$, of the counterparty risk contagion quadrant $N^v \times N^v$, of the overlapping portfolio contagion quadrant $N^v \times N^l$, and of the leverage targeting contagion quadrant $N^l \times N^v$.

S.7.1 Simplified Shock Transition Matrix

We first derive the simplified shock transition matrix to which systems converge as $N^s/N, N^d/N, N \to \infty$, after which we discuss how to reduce the resulting shock transition matrix to the 2×2 representative matrix.

- When $N^s/N \to \infty$ for all securities s, the market cap of each security s is distributed homogeneously over all institutions:

$$\lim_{N^s/N \to \infty} \frac{N_i^s}{N^s} = \mathbb{E}\left(\frac{N_i^s}{N^s}\right) = \frac{1}{N}, \tag{S.5}$$

  where $\mathbb{E}(\dots)$ denotes the expectation. All non-sink institutions that do not provide short-term lending have security $\hat{s}$ at the top of their pecking order, which is the most liquid security among the N^w distinct securities. The price-impact factor of security $\hat{s}$ is denoted as μ

(i.e. without any subscript).

- When $N^d/N, N \to \infty$, each leveraged institution's debt is distributed equally over all $N-1$ other institutions:

$$\lim_{N^d/N, N\to\infty} \frac{N^d_{ji}}{N^d_i} = \lim_{N\to\infty} \mathbb{E}\left(\frac{N^d_{ji}}{N^d_i}\right) = \lim_{N\to\infty} \frac{1}{N-1} = \frac{1}{N}. \tag{S.6}$$

The distribution of equities E_i that solves the balance sheet identity (23) and equations (S.5) and (S.6) is $E_i\left(1+\lambda_i\right) = E_j\left(1+\lambda_j\right)$ for any institutions i and j, where $\lambda_i = 0$ if institution i is a valuation sink and $\lambda_i = \lambda$ otherwise. That is, all leveraged institutions have the same equity and debt, and all valuation sinks have the same equity, which is equal to the sum of the equity and debt of any leveraged institution.

As equations (S.5) and (S.6) tell us how securities and total debt are distributed (and hence how overlapping portfolio contagion and counterparty risk contagion are distributed), let us now consider how short-term lending is distributed:

When $N^d/N, N \to \infty$, the fraction of short-term lender i's total short-term lending S_i provided to any leveraged institution j is equal to

$$\begin{aligned}
\lim_{N^d/N, N\to\infty} \frac{S_{ij}}{S_i} &= \lim_{N^d/N, N\to\infty} \frac{\frac{N^d_{ij}}{N^d_j} D_j}{\sum_{k=1}^{N^v} \frac{N^d_{ik}}{N^d_k} D_k} = \lim_{N^d/N, N\to\infty} \frac{\frac{N^d_{ij}}{N^d_j}}{\sum_{k=1}^{N^v} \frac{N^d_{ik}}{N^d_k}} \\
&= \lim_{N^d/N, N\to\infty} \frac{N^d_{ij}}{\sum_{k=1}^{N^v} N^d_{ik}} = \lim_{N^d/N, N\to\infty} \frac{\frac{N^d}{N^v-1}}{N^d} = \lim_{N\to\infty} \frac{1}{N^v-1} = \frac{1}{N^v},
\end{aligned} \tag{S.7}$$

where k runs over all leveraged institutions, and we have used that:

- When $N^s/N, N^d/N, N \to \infty$, all N^v leveraged institutions have the same debt D (as discussed above).

- $\lim_{N^d/N\to\infty} N_k^d = \mathbb{E}\left(N_k^d\right) = \frac{N^d N}{N^v}$ is the same for all leveraged institutions k (including institution j).

- The number of loans any leveraged institution i provides to another leveraged institution j is $\lim_{N^d/N\to\infty} N_{ij}^d = \mathbb{E}\left(N_{ij}^d\right) = \frac{N^d}{N^v-1}$, as institution i cannot lend to itself.

From equation (S.7) follows that the funding contagion transmission of a (non-sink) short-term lender, as given by the corresponding column in the simplified shock transition matrix, is equal to

$$\left[\frac{1}{N^v}, \ldots, \frac{1}{N^v}, 0, \frac{1}{N^v}, \ldots, \frac{1}{N^v}, 0, \ldots, 0\right]^T, \tag{S.8}$$

where for institution i, the i^{th} entry is zero (no lending to itself) and the last N^v terms are zero, which corresponds to the institution's non-transmission of overlapping portfolio contagion.

When $N \to \infty$, the institutions become a continuum and shock transmission to individual institutions vanishes. Hence, for $N \to \infty$, the funding contagion transmission vector reduces to[1]

$$\left[\frac{1}{N^v}, \ldots, \frac{1}{N^v}, 0, \ldots, 0\right]^T. \tag{S.9}$$

[1] Formally, the difference between S.8 and S.9 vanishes in the limit $N \to \infty$; $\lim_{N\to\infty} \left\| \left[\frac{1}{N^v}, \ldots, \frac{1}{N^v}, 0, \ldots, 0\right]^T - \left[\frac{1}{N^v-1}, \ldots, \frac{1}{N^v-1}, 0, \frac{1}{N^v-1}, \ldots, \frac{1}{N^v-1}, 0, \ldots, 0\right]^T \right\| = 0$.

such that each (non-sink) short-term lender's funding contagion is distributed homogeneously over the continuum of leveraged institutions.

From equation (S.6), it follows that for each passively leveraged institution, the counterparty risk contagion transmission as given by the corresponding column of the simplified shock transition matrix is equal to

$$\left[0, \ldots, 0, \frac{\delta\lambda}{N}, \ldots, \frac{\delta\lambda}{N}, 0, \frac{\delta\lambda}{N}, \ldots, \frac{\delta\lambda}{N}\right]^T, \tag{S.10}$$

where for institution i, the i^{th} entry is zero (no debt to itself), and the first N^l entries of the vector are zero, which corresponds to the institution's non-transmission of leverage targeting contagion.

Similar to funding contagion, when $N \to \infty$, the counterparty risk contagion transmission vector reduces to

$$\left[0, \ldots, 0, \frac{\delta\lambda}{N}, \ldots, \frac{\delta\lambda}{N}\right]^T. \tag{S.11}$$

such that each passively leveraged institution's counterparty risk contagion is distributed homogeneously over the continuum of institutions.

Lastly, from equation (S.5), the overlapping portfolio contagion shock transmission vector for any non-sink institution that does not provide short-term lending is equal to

$$\left[0, \ldots, 0, \frac{\mu}{N}, \ldots, \frac{\mu}{N}\right]^T, \tag{S.12}$$

where the first N^l terms are zero, which corresponds to the institution's non-transmission of funding contagion. Hence, the overlapping portfolio contagion transmitted by any non-sink institution that does not provide short-term lending is distributed homogeneously over the continuum of institutions.

From equations (S.9), (S.11) and (S.12), we find that for $N^s/N, N^d/N, N \to \infty$, the simplified shock transition matrix is given by

$$\left[\begin{array}{ccc||ccccccc}
I_1^f \frac{1}{N^v} & \cdots & I_{N^l}^f \frac{1}{N^v} & I_1^\lambda \lambda & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & 0 & I_2^\lambda \lambda & \ddots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \ddots & \vdots & \vdots & \ddots & \ddots & 0 & \vdots & \ddots & \vdots \\
I_1^f \frac{1}{N^v} & \cdots & I_{N^l}^f \frac{1}{N^v} & 0 & \cdots & 0 & I_{N^l}^\lambda \lambda & 0 & \cdots & 0 \\
\hline\hline
\left(1 - I_1^f\right) \frac{\mu}{N} & \cdots & \left(1 - I_{N^l}^f\right) \frac{\mu}{N} & (1 - I_1^\lambda) \frac{\delta\lambda}{N} & \cdots & \cdots & \left(1 - I_{N^l}^\lambda\right) \frac{\delta\lambda}{N} & \cdots & \cdots & (1 - I_{N^v}^\lambda) \frac{\delta\lambda}{N} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots \\
\left(1 - I_1^f\right) \frac{\mu}{N} & \cdots & \left(1 - I_{N^l}^f\right) \frac{\mu}{N} & (1 - I_1^\lambda) \frac{\delta\lambda}{N} & \cdots & \cdots & \left(1 - I_{N^l}^\lambda\right) \frac{\delta\lambda}{N} & \cdots & \cdots & (1 - I_{N^v}^\lambda) \frac{\delta\lambda}{N}
\end{array}\right]. \tag{S.13}$$

where $I_i^f = 1$ if institution i is a short-term lender and $I_i^f = 0$ otherwise, and $I_i^\lambda = 1$ if institution i has a leverage target and $I_i^\lambda = 0$ otherwise.

S.7.2 Reduction to Representative Matrix

We now show that the simplified shock transition matrix in equation (S.13) can be represented by a 2×2 matrix with largest eigenvalue identical to that of the shock transition matrix in equation (S.13). We do so by showing that system is uniquely determined by the dynamics of the aggregate liquidity and valuation shocks x_t^l and x_t^v.

Let $x_{t,i}^l$ denote the liquidity shock received by (non-sink) institution i at time t, such that $\vec{x}_t^l = \left[x_{t,1}^l, \ldots, x_{t,N^l}^l\right]^T$ and let $x_{t,j}^v$ be the valuation shock received by (leveraged) institution j at time t, such that $\vec{x}_t^v = \left[x_{t,1}^v, \ldots, x_{t,N^v}^v\right]^T$.

Furthermore, let $x_t^l = \sum_{i=1}^{N^l} x_{t,i}^l$ be the aggregate liquidity shock received by all non-sink institutions at time t and $x_t^v = \sum_{i=1}^{N^v} x_{t,i}^v$ be the aggregate valuation shock received by all leveraged institutions at time t. Lastly, at time t, let the fraction of the aggregate liquidity shock x_t^l received by non-sink institutions with short-term lending be denoted as $\hat{F}_t = \sum_{i=1}^{N^l} I_i^f x_{t,i}^l / x_t^l$, such that $(1 - \hat{F}_t) = \sum_{i=1}^{N^l} \left(1 - I_i^f\right) x_{t,i}^l / x_t^l$, and let the fraction of the aggregate valuation shock x_t^v received by leverage targeting institutions be denoted as $\hat{\Lambda}_t = \sum_{i=1}^{N^v} I_i^\lambda x_{t,i}^v / x_t^v$, such that $(1 - \hat{\Lambda}_t) = \sum_{i=1}^{N^v} \left(1 - I_i^\lambda\right) x_{t,i}^v / x_t^v$. We use the following properties throughout the derivation:

$$\frac{N^l}{N^v} = 1 - \phi_l, \tag{S.14}$$

$$\frac{\sum_{i=1}^{N^v} I_i^\lambda}{N^v} = \Lambda, \tag{S.15}$$

$$\lim_{N\to\infty} \frac{\sum_{i=1}^{N^l} I_i^f}{N^l} = F, \tag{S.16}$$

$$\lim_{N\to\infty} \frac{\sum_{i=1}^{N^l} I_i^f}{N^v} = \lim_{N\to\infty} \frac{(1-\phi_l)\sum_{i=1}^{N^l} I_i^f}{N^l} = (1-\phi_l)F, \tag{S.17}$$

$$\lim_{N\to\infty} \frac{\sum_{i=1}^{N^l} I_i^\lambda}{N^v} = \lim_{N\to\infty} \frac{(1-\phi_l)\sum_{i=1}^{N^l} I_i^\lambda}{N^l} = (1-\phi_l)\Lambda \tag{S.18}$$

$$\lim_{N\to\infty} \sum_{i=1}^{N^l} I_i^f I_i^\lambda = \sum_i^{N^l} \mathbb{E}\left(I_i^f I_i^\lambda\right) = \sum_i^{N^l} \mathbb{E}\left(I_i^f\right) \mathbb{E}\left(I_i^\lambda\right) = F\Lambda \sum_i^{N^l} 1 = F\Lambda N^l \tag{S.19}$$

The first identity is simply a restatement of the fact that we designate a fraction ϕ_l of leveraged institution as liquidity sinks. The second is a restatement of the definition that the fraction of leverage targeters Λ is equal to the fraction of leveraged institutions that have a leverage target. The third and fourth identities use that as $N \to \infty$, the fraction of institutions that provide short-term lending is equal to the fraction of non-sink institutions that provide short-term lending (because sinks and short-term lenders are designated independently). Similarly, the fifth identity uses that as $N \to \infty$, the fraction of non-sink institutions that have a leverage target is equal to Λ (because liquidity sinks and leverage strategies are designated independently). Lastly, the sixth identity gives the number of non-sink short-term lenders that are leverage targeting and follows from the fact that short-term lenders and leverage targeting institutions are designated independently.

Plugging the simplified shock transition matrix in equation (S.13) into equation (2) yields

$$\vec{x}_{t+1} = A\vec{x}_t = A\begin{bmatrix}\vec{x}_t^l \\ \vec{x}_t^v\end{bmatrix} = \begin{bmatrix} \frac{1}{N^v}\sum_{i=1}^{N^l} I_i^f x_{t,i}^l + \lambda I_1^\lambda x_{t,1}^v \\ \vdots \\ \frac{1}{N^v}\sum_{i=1}^{N^l} I_i^f x_{t,i}^l + \lambda I_{N^l}^\lambda x_{t,N^l}^v \\ \frac{\mu}{N}\sum_{i=1}^{N^l}\left(1-I_i^f\right)x_{t,i}^l + \frac{\delta\lambda}{N}\sum_{j=1}^{N^v}\left(1-I_j^\lambda\right)x_{t,j}^v \\ \vdots \\ \frac{\mu}{N}\sum_{i=1}^{N^l}\left(1-I_i^f\right)x_{t,i}^l + \frac{\delta\lambda}{N}\sum_{j=1}^{N^v}\left(1-I_j^\lambda\right)x_{t,j}^v \end{bmatrix} \tag{S.20}$$

$$= \begin{bmatrix} \frac{1}{N^v}\hat{F}_t x_t^l + \lambda I_1^\lambda x_{t,1}^v \\ \vdots \\ \frac{1}{N^v}\hat{F}_t x^l + \lambda I_{N^l}^\lambda x_{t,N^l}^v \\ \frac{\mu}{N}(1-\hat{F}_t)x_t^l + \frac{\delta\lambda}{N}(1-\hat{\Lambda}_t)x_t^v \\ \vdots \\ \frac{\mu}{N}(1-\hat{F}_t)x_t^l + \frac{\delta\lambda}{N}(1-\hat{\Lambda}_t)x_t^v \end{bmatrix} \tag{S.21}$$

Hence, at time $t+1$, we have for any non-sink institution i that

$$x_{t+1,i}^l = \frac{\hat{F}_t}{N^v}x_t^l + \lambda I_i^\lambda x_{t,i}^v, \tag{S.22}$$

and for any leveraged institution i that

$$x_{t+1,i}^v = \frac{(1-\hat{F}_t)\mu}{N}x_t^l + \frac{(1-\hat{\Lambda}_t)\delta\lambda}{N}x_t^v. \tag{S.23}$$

Equation (S.23) shows that $x_{t+1,i}^v$ is the same for any (leveraged) institution i, because the right-hand side of equation (S.23) does not depend on i.

The aggregate valuation shock at time $t+1$ is given by

$$x^v_{t+1} = \sum_{i=1}^{N^v} x^v_{t+1,i} = (1-\hat{F}_t)(1-\phi_v)\mu x^l_t + (1-\hat{\Lambda}_t)(1-\phi_v)\delta\lambda x^v_t \tag{S.24}$$

and hence we have for any leveraged institution i that

$$x^v_{t+1,i} = \frac{x^v_{t+1}}{N^v}. \tag{S.25}$$

From $x^v_{t+1,i} = x^v_{t+1}/N^v$ follows that

$$\hat{\Lambda}_{t+1} = \frac{\sum_{i=1}^{N^v} I^\lambda_i x^v_{t+1,i}}{x^v_{t+1}} = \frac{\sum_{i=1}^{N^v} I^\lambda_i \frac{x^v_{t+1}}{N^v}}{x^v_{t+1}} = \frac{\sum_{i=1}^{N^v} I^\lambda_i}{N^v} = \Lambda. \tag{S.26}$$

Furthermore, from $x^v_{t+1,i} = x^v_{t+1}/N^v$ also follows that, at time $t+2$,

$$x^l_{t+2,i} = \frac{\hat{F}_{t+1}}{N^v} x^l_{t+1} + \frac{\lambda I^\lambda_i}{N^v} x^v_{t+1}, \tag{S.27}$$

$$x^l_{t+2} = \hat{F}_{t+1}(1-\phi_l)x^l_{t+1} + \lambda x^v_{t+1} \sum_{i=1}^{N^l} \frac{I^\lambda_i}{N^v} = \hat{F}_{t+1}(1-\phi_l)x^l_{t+1} + \Lambda(1-\phi_l)\lambda x^v_{t+1}. \tag{S.28}$$

Using equations (S.27) and (S.28), we find that

$$\hat{F}_{t+2} = \frac{\sum_{i=1}^{N^l} I_i^f x_{t+2,i}^l}{x_{t+2}^l} = \frac{\sum_{i=1}^{N^l} I_i^f \left(\frac{\hat{F}_{t+1}}{N^v} x_{t+1}^l + \frac{\lambda I_i^\lambda}{N^v} x_{t+1}^v \right)}{\hat{F}_{t+1}(1-\phi_l)x_{t+1}^l + \Lambda(1-\phi_l)\lambda x_{t+1}^v} \tag{S.29}$$

$$= \frac{\hat{F}_{t+1} x_{t+1}^l \frac{\sum_{i=1}^{N^l} I_i^f}{N^v} + \lambda x_{t+1}^v \frac{\sum_{i=1}^{N^l} I_i^f I_i^\lambda}{N^v}}{\hat{F}_{t+1}(1-\phi_l)x_{t+1}^l + \Lambda(1-\phi_l)\lambda x_{t+1}^v} \tag{S.30}$$

$$= \frac{F\left(\hat{F}_{t+1}(1-\phi_l)x_{t+1}^l + \Lambda(1-\phi_l)\lambda x_{t+1}^v\right)}{\hat{F}_{t+1}(1-\phi_l)x_{t+1}^l + \Lambda(1-\phi_l)\lambda x_{t+1}^v} = F, \tag{S.31}$$

Equations (S.23) and (S.27) do not depend on individual shocks but only on the aggregate liquidity and valuation shocks. Therefore, we find that for $t > 1$ (where the initial exogenous shock occurs at $t = 0$)[2], the system's shock propagation is uniquely determined by the dynamics of x_t^l and x_t^v, which we find by plugging $\hat{F}_t = F$ and $\hat{\Lambda}_t = \Lambda$ into equations (S.24) and (S.28):

$$\begin{aligned} x_{t+1}^l &= F(1-\phi_l)x_t^l + \Lambda(1-\phi_l)\lambda x_t^v \\ x_{t+1}^v &= (1-F)(1-\phi_v)\mu x_t^l + (1-\Lambda)(1-\phi_v)\delta\lambda \end{aligned} \tag{S.32}$$

Equation (S.32) can be written in matrix form as

[2] This limitation is not an artifact of the derivation but an actual constraint: When $\vec{x}_{t=0}$ consists of a single valuation shock to a leverage targeting institution i, $\vec{x}_{t=1}$ consists of a single liquidity shock to the same institution i, so equation (S.33) does not hold for $t = 0$. Depending on whether or not institution i has made short-term loans, institution i transmits either a pure funding or pure overlapping portfolio contagion shock (so either $x_{t=2}^l = 0$ or $x_{t=2}^v = 0$) so equation (S.33) also does not hold for $t = 1$. Because the funding or overlapping portfolio contagion shock is distributed homogeneously over all institutions, equation (S.33) holds from $t = 2$ onward. However, this $\vec{x}_{t=0}$ is not an eigenvector of A, because $\vec{x}_{t=1}$ is orthogonal to $\vec{x}_{t=0}$). When $\vec{x}_{t=0}$ *is* an eigenvector of A, (S.33) holds for $t = 1$.

$$\begin{bmatrix} x^l_{t+1} \\ x^v_{t+1} \end{bmatrix} = \left[\begin{array}{c||c} F(1-\phi_l) & \Lambda(1-\phi_l)\lambda \\ \hline\hline (1-F)(1-\phi_v)\mu & (1-\Lambda)(1-\phi_v)\delta\lambda \end{array}\right] \begin{bmatrix} x^l_t \\ x^v_t \end{bmatrix} \tag{S.33}$$

such that the representative matrix is given by

$$\hat{A} = \left[\begin{array}{c||c} F(1-\phi_l) & \Lambda(1-\phi_l)\lambda \\ \hline\hline (1-F)(1-\phi_v)\mu & (1-\Lambda)(1-\phi_v)\delta\lambda \end{array}\right]. \tag{S.34}$$

Let ν be the largest eigenvalue of the representative matrix and $\vec{v}$ the corresponding eigenvector, such that

$$\hat{A}\vec{v} = \hat{A} \begin{bmatrix} \vec{v}^l \\ \vec{v}^v \end{bmatrix} = \nu \begin{bmatrix} \vec{v}^l \\ \vec{v}^v \end{bmatrix}. \tag{S.35}$$

When we rewrite equations (S.23) and (S.27) as a matrix-vector product and use that $\hat{F}_t = F$ and $\hat{\Lambda}_t = \Lambda$ for $t > 1$:

$$\vec{x}_{t+1} = \begin{bmatrix} \frac{F(1+h)}{N^v} & \frac{\lambda I_i^\lambda}{N^v} \\ \vdots & \vdots \\ \frac{F(1+h)}{N^v} & \frac{\lambda I_i^\lambda}{N^v} \\ \frac{(1-F)\mu}{N} & \frac{(1-\Lambda)\delta\lambda}{N} \\ \vdots & \vdots \\ \frac{(1-F)\mu}{N} & \frac{(1-\Lambda)\delta\lambda}{N} \end{bmatrix} \begin{bmatrix} x^l_t \\ x^v_t \end{bmatrix} \tag{S.36}$$

and because the matrix is constant, we see that $\vec{x}_t$ grows by ν when $\left[x^l_t, x^v_t\right]^T$ grows by ν. Therefore, ν is also the largest eigenvalue of the (full) shock transition matrix A. $\square$

S.8 Eigenvalue Dependence on Cycles and Time Dynamics

We begin by giving a short proof that the shock transition matrix' largest eigenvalue, when is equal to one, is independent of the relative speeds at which the various channels act. The proof is followed by a more in-depth discussion of how the largest eigenvalue is determined by the cycles in the network.

S.8.1 Short Proof

When the largest eigenvalue $\nu = 1$, the corresponding eigenvector is invariant under the shock transition matrix' dynamic,

$$\vec{v}_{t+1} = A\vec{v}_t = \nu\vec{v}_t = \vec{v}_t. \tag{S.37}$$

Hence, for any element v_k (corresponding to the network's k^{th} node) of the eigenvector $\vec{v}$, we have that

$$v_{k,t+1} = v_{k,t}. \tag{S.38}$$

From the matrix-vector product, we know that

$$v_{k,t+1} = \sum_{y \in \mathcal{A}} w_{yk} v_{y,t}, \tag{S.39}$$

where $\mathcal{A}$ denotes the network's set of nodes and $w_{yk} = A_{ky}$ denotes the weight of the edge from node y to node k. Hence,

$$v_{k,t} = \sum_{y \in \mathcal{A}} w_{ik} v_{y,t} \tag{S.40}$$

Let us now take a specific pair of nodes i, k. We add a "dummy node" *between* nodes i and k, by redirecting i's edge to node k to node j and adding an edge with weight equal to one from node j to node k. Using $\hat{w}_{xy}$ to denote edges in the new network, we have that $\hat{w}_{ij} = w_{ik}$, $\hat{w}_{ik} = 0$, $\hat{w}_{jk} = 1$, and $\hat{w}_{yx} = w_{yx}$ for any pair of nodes yx other than the pairs ij, ik and jk. Hence, any shock that was previously transmitted directly from i to k is now delayed by one iteration before arriving at node k, while the shock's magnitude is unaffected.

Compared to $\vec{v}$, the new network's eigenvector $\hat{\vec{v}}$ has an additional entry, $\hat{v}_j$. For all $y \neq j$ we set $\hat{v}_{y,t} = v_{y,t}$, and since $\hat{v}_{j,t+1} = \hat{w}_{ij} \hat{v}_{i,t} = w_{ik} v_{i,t}$, we set $\hat{v}_{j,t} = w_{ik} v_{i,t}$. Using $\hat{\mathcal{A}}$ to denote the new network's set of nodes, we immediately see that

$$\hat{v}_{k,t+1} = \sum_{y \in \hat{\mathcal{A}}} \hat{w}_{yk} \hat{v}_{y,t} = \sum_{y \in \mathcal{A}} w_{yk} v_{y,t} = v_{k,t} = \hat{v}_{k,t}, \tag{S.41}$$

so the invariance in equation (S.38) is conserved (as well as the invariance of the entire eigenvector (S.37), as the rest of the network remains unchanged). Hence, when the largest eigenvalue is equal to one, we can add a dummy node that slows down shock transmission, without affecting the invariance of the corresponding eigenvector's invariance under the shock transition matrix' dynamic. In principle, we can add any number of dummy nodes to "tune" the relative speeds at which contagion channels operate with any desired granularity, without affecting the network's stability. Thus, when the largest eigenvalue is equal to one, the network's stability is independent of the relative speeds at which contagion channels operate.

S.8.2 Cyclic Paths

The largest eigenvalue of the adjacency matrix of a directed network (with nonnegative weights) is determined by the cycles present in the network. For the shock transition matrix, these correspond to circular exposure-relationships between institutions. We will now formally define cycles and discuss how they determine the largest eigenvalue.

We say that a node's out-edges *depart* from the node and a node's in-edges *arrive* at the node. A *path* in a directed network be an ordered sequence of edges of which each edge but the first departs from the node the previous edge in the sequence arrived at. The node from which the first edge departs is the *starting node* and the node the last edge arrives at is the *ending node*. When the starting and ending node coincide, the path is a *cycle*. A *direct cycle* is a cyclic path consisting of a single edge, which directly returns to the starting node.

A set of nodes in a directed network is *strongly connected* when each node in the set can be reached from every other node in the set (by following the directed edges). A *Strongly Connected Component* (SSC) of a network is a strongly connected set of nodes which is *maximal*; adding any additional node in the network to the set breaks the set's strongly-connectedness.

Any two nodes i and j in an SSC are connected by at least one cycle, as at least one path from i to j and one path from j to i exists.

We define the set of *simple paths* from node x to node y, $x \neq y$, as the set of all paths from x to y that include node x only once (as the starting node). Simple paths are allowed to include the ending node y multiple times. The set of *simple cycles* of node i consists of all cyclic paths that depart from i (as the starting node) and arrive at i only once (as the ending node). Hence, node i's set of simple cycles consists of all cyclic path that end upon returning to node i for the first time.

S.8.3 Directed Acyclic Networks

A directed acyclic network (DAG) is a directed network that does not contain any cycles (of all possible paths in the network, none are cyclic). A DAG is a *directed tree*; nodes can be ordered vertically with any node only having out-edges to nodes lower in the vertical order. Equivalently, we can order the DAG's adjacency matrix such that it only has non-zero entries below the diagonal; as a row in the adjacency matrix gives the corresponding node's in-edges, we can order the matrix such that any node q only has in-edges from nodes whose rows are above node q's row. This vertical order of rows corresponds to the vertical order of the DAG's tree representation. The diagonal entries of a triangular matrix give the matrix' eigenvalue spectrum. Therefore, a DAG does not have any non-zero eigenvalues. The intuition behind this is as follows: A DAG network consisting of N nodes cannot paths of N edges or more, because that would require visiting a node twice, which constitutes a cycle. Therefore, when A is DAG, the sequence of shock-vectors generated according to

$$\vec{x}_{t+1} = A\vec{x}_t \tag{S.42}$$

must terminate after at most $N - 1$ steps ($\vec{x}_{t+N} = 0$). Conversely, the eigenvalue equation

$$\vec{x}_{t+1} = A\vec{x}_t = \nu\vec{x}_t \tag{S.43}$$

defines an infinite sequence of (nonzero) eigenvectors when the largest eigenvalue ν is nonzero (and $\vec{x}_t$ has a nonzero component in the direct of $\nu' s$ eigenvector). Hence, a DAG cannot have any nonzero eigenvalues as it can only propagate a shock (vector) a finite number of times before it must necessarily terminate.

S.8.4 Acyclic Paths

Acyclic nodes and paths are nodes and paths that are not included in any cycle. Any directed network can be represented as a *directed block tree.* The blocks correspond to the network's SSCs, where any acyclic node or node only included in a direct cycle constitutes its own SSC. Cycles that include nodes from multiple SSCs cannot exist or the SSCs would not be maximal. Therefore, we can order the SSCs vertically such that nodes in each SSC only have out-edges to other nodes in the SSC and to nodes in SSCs lower in the vertical order.

Equivalently, we can order any directed network's adjacency matrix to be lower block triangular, with its SSCs on its diagonal in the (vertical) order of the directed block tree. Hence, all entries above the (block) diagonal are zero, as these correspond to nodes' out-edges to SSCs higher in the vertical order of the directed block tree.

The eigenvalue spectrum of a block-triangular matrix is given by the union of the diagonal blocks' eigenvalue spectra. As acyclic nodes appear as (1×1 blocks of) zeros on the diagonal, the network's largest eigenvalue is determined by its SSCs. Hence, acyclic nodes and edges do not affect the network's eigenvalues. In other words, both sinks and nodes with no in-edges can be excluded from the shock transition matrix without affecting its largest eigenvalues.

S.8.5 Edge-Weight-Products

We call the product of the weights of the edges that make up a path the path's edge-weight-product. The weights of the (directed) edges are given by the shock transition matrix' entries; the weight of a directed edge gives the size of the shock transmitted to the receiving node at $t + 1$, relative to the shock at the transmitting node at time t. Hence, the edge-weight-product of a path gives the factor by which a shock is amplified or damped when it traverses the path (from starting to ending node). Specifically, a cycle's edge-weight-product gives the factor by which a propagating shock is amplified or damped each time it laps the cycle.

S.8.6 Edge-Weight-Products and Eigenvalues

We now elucidate the relation between an SSC's largest eigenvalue and its cycles' lengths and edge-weight-products. Before presenting the general case, we first discuss two simple examples.

Single-Cycle Example

Consider an SSC that consists of a single, simple cycle (for any choice of reference node), as illustrated in Figure S.10a. We denote the edge-weight-product of this cycle as ϵ and its length (number of edges) as K. Hence, a shock is amplified or damped by a factor ϵ every time it laps the cycle in K time steps. We now show that the largest eigenvalue is equal to $\nu = \sqrt[K]{\epsilon}$ and that the corresponding eigenvector is amplified or damped by a factor $\sqrt[K]{\epsilon}$ every iteration. We do this by constructing an eigenvector $\vec{v}$ corresponding to the largest eigenvalue ν: as any scalar multiple of an eigenvector is also an eigenvector of the same eigenvalue, we are free to choose the first node's entry in the eigenvector $\vec{v}_{t,1} = 1$, which fixes the other entries of the eigenvector.

Let w_j denote the weight of the edge from node j to node $j+1$ in the cycle. As node 1's value is equal to 1 at time t, node 2's value at time $t+1$ is equal to $\vec{v}_{t+1,2} = w_1\vec{v}_{t,1}$. Given that $\vec{v}_{t+1,2} = \nu\vec{v}_{t,2}$, we must have that

$$\vec{v}_{t,2} = \frac{w_1}{\nu}\vec{v}_{t,1} = \frac{w_1}{\nu}. \tag{S.44}$$

Therefore,

$$\vec{v}_{t+j-1,j} = w_{j-1}\vec{v}_{t+j-2,j-1} = w_{j-1}w_{j-2}\vec{v}_{t+j-3,j-2} = ... = \prod_{i=1}^{j-1} w_i \vec{v}_{t,1} = \prod_{i=1}^{j-1} w_i, \tag{S.45}$$

and at time t

$$\vec{v}_{t,j} = \frac{\prod_{i=1}^{j-1} w_i}{\nu^{j-1}}. \tag{S.46}$$

From setting $j = K$ follows

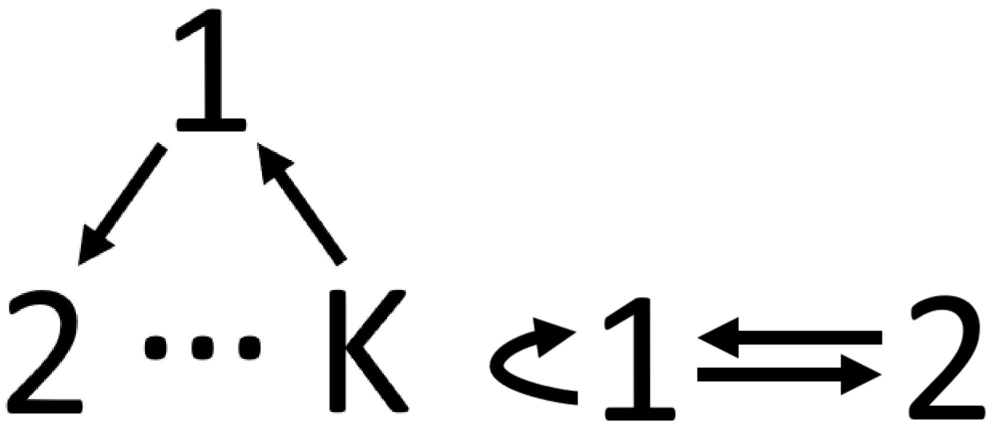
$$\vec{v}_{t+1,1} = w_K \frac{\prod_{i=1}^{K-1} w_i}{\nu^{j-1}} = \frac{\prod_{i=1}^{K} w_i}{\nu^{j-1}}, \tag{S.47}$$

$$\vec{v}_{t,1} = \frac{\prod_{i=1}^{K} w_i}{\nu^{K}} = 1. \tag{S.48}$$

Thus, $\nu = \sqrt[K]{\epsilon}$, with $\epsilon = \prod_{i=1}^{K} w_i$. Specifically, when $\nu = 1$, the length of the cycle K drops out of the equation and we find $\epsilon = 1$. Thus, whether the largest eigenvalue is equal to one depends only on cycles' edge-weight-products and not their lengths.

Multi-Cycle Example

We now consider an example of an SSC consisting of two nodes that includes multiple simple cycles. Nodes 1 and 2 both have an edge to the other node, and node 1 also has an edge to itself (a direct cycle), as shown in Figure S.10b. The network has edges with weights w_{11}, w_{12} and w_{21}, where w_{xy} denotes the weight of the edge from x to y.



(a) (b)

Figure S.10: **Examples of simple networks**. We plot a network consisting of a single loop of K nodes in (a) and a network consisting of two nodes that includes multiple loops in (b).

Using node 1 as our reference node, the network has two simple cycles: The direct cycle $1 \to 1$ and the cycle $1 \to 2 \to 1$ of length 2. We set $\vec{v}_{t,1} = 1$, so $\vec{v}_{t,2} = \frac{w_{12}}{\nu}$. Consequently, we find

$$\vec{v}_{t+1,1} = w_{11}\vec{v}_{t,1} + w_{21}\vec{v}_{t,2} = w_{11} + \frac{w_{12}}{\nu}, \tag{S.49}$$

and at time t

$$\vec{v}_{t,1} = \frac{w_{11}}{\nu} + \frac{w_{12}}{\nu^2} = 1. \tag{S.50}$$

Plugging $\nu = 1$ into equation (S.50) yields $w_{11} + w_{12}w_{21} = 1$; the sum of the cycles' edge-weight-products must be equal to one for the (largest) eigenvalue to be equal to one. Note that this is independent of these cycles' lengths.

We find the same result by choosing $N_2 = 1$ as our reference node. The corresponding set of simple cycles includes an infinite number of cycles,

$2 \to 1 \to 2$, $2 \to 1 \to 1 \to 2$, and $2 \to 1 \to ... \to 1 \to 2$. Setting $\vec{v}_{t,2}$=1 yields

$$\vec{v}_{t+1,1} = w_{21}\vec{v}_{t,2} + w_{11}\vec{v}_{t,1} = w_{21} + w_{11}\vec{v}_{t,1}, \tag{S.51}$$

and at time t

$$\vec{v}_{t,1} = \frac{w_{21}}{\nu} + \frac{w_{11}\vec{v}_{t,1}}{\nu}. \tag{S.52}$$

It is to straightforward to reduce equation (S.52) to equation (S.50), by using $\vec{v}_{t+1,2} = w_{12}\vec{v}_{t,1}$, which implies

$$\vec{v}_{t,2} = w_{12}\vec{v}_{t,1}/\nu = 1. \tag{S.53}$$

However, for illustrative purposes, we take an alternative, recursive approach:

$$\vec{v}_{t,1} = \frac{w_{21}}{\nu} + \frac{w_{11}\left(\frac{w_{21}}{\nu} + \frac{w_{11}\vec{v}_{t,1}}{\nu}\right)}{\nu} = \frac{w_{21}}{\nu} + \frac{w_{11}\left(\frac{w_{21}}{\nu} + \frac{w_{11}\left(\frac{w_{21}}{\nu} + \frac{w_{11}\vec{v}_{t,1}}{\nu}\right)}{\nu}\right)}{\nu}, \tag{S.54}$$

$$\vec{v}_{t,1} = \sum_{i=0}^{\infty} \frac{w_{21}}{\nu}\left(\frac{w_{11}}{\nu}\right)^i + \lim_{j\to\infty}\left(\frac{w_{11}}{\nu}\right)^j \vec{v}_{t,1}, \tag{S.55}$$

where $\frac{w_{21}}{\nu} < 1$, because

$$\vec{v}_{t+1,1} = \nu\vec{v}_{t,1} = w_{11}\vec{v}_{t,1} + w_{21}\vec{v}_{t,2} = w_{11}\vec{v}_{t,1} + w_{21}, \tag{S.56}$$

(all terms are positive, because $\vec{v}_{t,2}$ is positive and all edge-weight are non-negative). Therefore, $\lim_{j\to\infty} (w_{11}/\nu)^j \vec{v}_{t,1} = 0$ and

$$\vec{v}_{t,1} = \sum_{i=0}^{\infty} \frac{w_{21}}{\nu} \left(\frac{w_{11}}{\nu}\right)^i . \tag{S.57}$$

Equation (S.57) sets $\vec{v}_{t,1}$ equal to the sum over all simple paths from node 2 to node 1, of each path's edge-weight-product divided by the largest eigenvalue ν raised to the power of the path's length. Equation (S.57) is also the result one finds by applying the following "power iteration" algorithm to finding $\vec{v}_{t,1}$:

1. Set $\vec{v}_{t,2} = 1$ and $\vec{v}_{t,1} = 0$
2. Set $\vec{v}_{t+1,1} = w_{11}\vec{v}_{t,1} + w_{21}\vec{v}_{t,2} = w_{11}\vec{v}_{t,1} + w_{21}$.
3. Set $\vec{v}_{t,1} = \vec{v}_{t+1,1}/\nu$.
4. Repeat steps 2. and 3. until convergence.

Hence, the algorithm is a power iteration[3] that uses the initial vector $\vec{x}_0 = [0,1]^T$ and has a "modified" normalization step, which divides the top entry of the vector by the largest eigenvalue ν and sets the bottom entry to one. As the vector converges to the eigenvector, this modified normalization step converges to dividing the vector (i.e. both the top and bottom entry) by the largest eigenvalue ν, which is the same normalization as "regular" power iteration converges to.

[3]Power iteration is an algorithm of finding the largest eigenvalue and corresponding eigenvector of a matrix by generating a sequence of vectors, where each subsequent vector is obtained by multiplying the previous vector with the matrix. Typically, each subsequent vector is normalized to prevent machine-precision issues when the largest eigenvalue is much greater or smaller than one (which causes subsequent vectors to grow or shrink rapidly).

From equations (S.53) and (S.57), we find

$$\vec{v}_{t,2} = \frac{w_{21}w_{12}}{\nu^2}\sum_{i=0}^{\infty}\left(\frac{w_{11}}{\nu}\right)^i = 1. \tag{S.58}$$

The RHS of equation (S.58) is the sum over all simple cycles with respect to node 2, of each cycle's edge-weight-product divided by ν raised to the power of the cycle's length. From the geometric series $\sum_{i=0}^{\infty} w_{11}/\nu = 1/(1 - w_{11}/\nu)$ follows straightforwardly that equation (S.58) reduces to equation (S.50).

Arbitrary Strongly Connected Component

We now consider an SSC with arbitrary topology (and with nonnegative edge-weights). Given that the SSC's largest eigenvalue is equal to ν, we construct the corresponding eigenvector. We choose a reference node i and set $\vec{v}_{t,i} = 1$. We then apply the power iteration described above: For any node $j \neq i$, $\vec{v}_{t,j}$ converges to the sum over all simple paths from node i to node j, of each path's edge-weight-product divided by ν raised to the power of the path's length:

$$\vec{v}_{t,j} = \sum_{p\in\mathcal{P}_{ij}} \frac{\epsilon_p}{\nu^{\eta_p}}, \tag{S.59}$$

where $\mathcal{P}_{ij}$ denotes the set of all simple paths from node i to node j, ϵ_p the path's edge-weight-product, and η_p the path's length.

The vector $\vec{v}_t$ we have constructed is an eigenvector of the largest eigenvalue ν, as it satisfies the eigenvalue equation: For any node $j \neq i$ in the SSC, let $\mathcal{Z}$ denote the set of node j's in-nodes (the nodes from which node

j's in-edges depart). The shocks transmitted by node j's in-nodes yield $\vec{v}_{t+1,j}$:

$$\vec{v}_{t+1,j} = \sum_{z\in\mathcal{Z}} w_{zj}\vec{v}_{t,z} = \sum_{z\in\mathcal{Z}} w_{zj} \sum_{p\in\mathcal{P}_{iz}} \frac{\epsilon_p}{\nu^{\eta_p}} = \sum_{p\in\mathcal{P}_{ij}} \frac{\epsilon_p}{\nu^{\eta_p-1}} = \nu\vec{v}_{t,j}, \tag{S.60}$$

where the last step follows from the fact that node j's in-nodes are the penultimate nodes in all paths that end at node j. The relation between the SSC's largest eigenvalue ν and the edge-weight products of the SSC's cycles follows from

$$\vec{v}_{t+1,i} = \sum_{z\in\mathcal{Z}} w_{zi}\vec{v}_{t,z} = \sum_{z\in\mathcal{Z}} w_{zi} \sum_{p\in\mathcal{P}_{iz}} \frac{\epsilon_p}{\nu^{\eta_p}} = \sum_{p\in\mathcal{C}_{i}} \frac{\epsilon_p}{\nu^{\eta_p-1}} = \nu, \tag{S.61}$$

where $\mathcal{C}_i$ denotes the set of simple cycles with respect to i and we have used that $\vec{v}_{t+1,i} = \nu\vec{v}_{t,i} = \nu$. Thus,

$$\sum_{p\in\mathcal{C}_i} \frac{\epsilon_p}{\nu^{\eta_p}} = 1. \tag{S.62}$$

Equation (S.62) shows the relation between an SSC's largest eigenvalue ν and the SSC's cycles: For any reference node i, the sum over all simple cycles with respect to i of each cycle's edge-weight-product divided by ν^{η_p} must equal one, where ν^{η_p} is the largest eigenvalue raised to the power of the path's length. Specifically, for the largest eigenvalue to equal one, the sum of all cycles' edge-weight-products must equal one, regardless of the cycles' lengths. Equation (S.62) shows that the largest eigenvalue ν is a monotonic increasing function of the edge-weights W (as the weights and largest eigenvalue are non-negative). Hence, ceterus paribus, equation

(S.62) specifies how large institutions' leverages may be without causing the system to be inherently unstable.

We now consider these insights in the context of the shock transition matrix' time dynamics: The longer a path is, the longer (i.e. the more iterations) it takes a shock to traverse the path. Furthermore, different contagion mechanisms operate on different timescales. Therefore, the time it takes a shock to traverse a path should also depend on which contagion mechanism each edge in the path corresponds to. Instead, in the shock transition matrix, each contagion mechanism acts equally fast. However, we can correct for time-differences between contagion mechanisms by adding intermediate "dummy" edges (and, correspondingly, "dummy nodes") with weight equal to one to increase the lengths of paths corresponding to "slow" contagion mechanisms. This increase the number of iterations it takes to traverse the path. Specifically, it changes cycles' lengths, but not their edge-weight-products. As the largest eigenvalue depends only on cycles' edge-weight-products (and not their lengths) for being equal to one, and we can implement realistic time dynamics without affect cycles' edge-weight-products (by changing only the cycles' lengths), we conclude that whether or not a system is inherently unstable is independent of the timescales the various contagion mechanisms operate on.